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Gibbs Delaunay Tessellations with Geometric Hardcore Conditions

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Abstract In this paper, we prove the existence of infinite Gibbs Delaunay tessellations on $\mathbb{R}^2$. The interaction depends on the local geometry of the tessellation. We introduce a geometric hardcore condition on small and large cells, consequently we can construct more regular infinite random Delaunay tessellations.

Keywords Stochastic geometry · Gibbs measure · Delaunay tessellation

1 Introduction

The importance of the Gibbs point process as a model in the statistical analysis of spatial point patterns is today widely recognized. Indeed, the class of Gibbs point processes is interesting because it allows one to introduce and study interactions between points through the modeling of an associated potential function. In [5–7] Bertin, Billiot and Drouilhet deal with Gibbs point processes where the interaction depends on the structure of the Delaunay tessellation. More precisely, they consider the Delaunay triangulation in the space $\mathbb{R}^2$ where the vertices are given by the point process and the interaction is built thanks to the triangles of this tessellation. It is a functional from the local geometry of the Delaunay tessellation. This produces, for the classical point of view, an infinite multibody interaction (see [9, 16, 17]). In [5], they study the finite volume case. They develop some algorithms to simulate these Gibbs tessellations and give some applications for the modeling of the cells of the prostatic tissue. In [6, 7], they prove the existence of some infinite Gibbs states. The results are proved in the classical context of the Gibbs point process using Preston's well-known Theorems (see [14]).

The concept of random simplicial complexes, including simplicial surfaces or networks, is widely developed in Mathematics and Physics (see [19, 21]). In this context of random

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geometry, Zessin in [21] proposes a purely geometric approach to Gibbs simplicial complexes, the objective being the problem of the pure quantum gravity (see [21] for many references). Typically, the interaction potential is a functional from the local intrinsic geometry defined in terms of invariants, such as the volume and the curvature. The main difficulty of such a theory is how to define this reference measure which should be a random simplicial complex without interaction. In the classical context of point processes, the reference measure is obviously the Poisson point process. But for the random simplicial complexes, the question is open. Moreover, even if we have a natural reference measure for random simplicial complexes, we have to describe the local laws of this measure given the outside configuration to obtain the classical DLR equations.

The objective of this paper is the development of a purely geometric description of the random Gibbs simplicial complexes, built on the Delaunay tessellations. Our work is a continuation of the results of Bertin, Billiot and Drouilhet in [5–7] in the context of the simplicial complexes inspired by Zessin. Indeed, we use his point of view to define the energy from the local intrinsic geometry of the tessellation. Moreover, we propose a new and more sophisticated description of the local densities which is more compatible with the geometry of the tessellation than in the classical case. Let us give an example to illustrate these notions. Let us suppose that the energy is concentrated on the triangles of the tessellation, so, for a tessellation Γ and for a bounded window Λ in $\mathbb{R}^2$ we define the energy of Γ in Λ

$$E_\Lambda(\Gamma) = \sum_{X\in\Gamma, X\cap\Lambda\neq\emptyset} V(X), \tag{1}$$

where the sum is for all triangles X in Γ such that X intersects Λ. $V(X)$ is the energy of triangle X. Now, the reference measure P is just the random Delaunay tessellation for which the random vertices are given by a Poisson process. The DLR equation on Λ is consequently obtained thanks to the local laws of P, given all the triangles outside of Λ or crossing the boundary, for which we add the classical density $\frac{1}{Z_\Lambda}e^{-E_\Lambda(\Gamma)}$. The DLR equations in [5–7] are different and simpler because they are in the context of classical Gibbs point processes. We think however that our approach is more natural for the physical point of view since our descriptions of the local laws are directly focused on the intrinsic geometry of the tessellation.

Thanks to this new approach, in Theorem 1 Sect. 3, we have improved a result in [6]. Indeed, using some ideas from Schreiber in [18] to control the convergence of the finite volume Gibbs tessellations and the locality of the tessellations, we relax an assumption of quasilocality for the interaction. In the classical context of Gibbs point processes, this assumption is natural because the interaction of a configuration goes to zero when the diameter goes to infinity. In the context of Gibbs Delaunay tessellations, it is not however natural because the energy of a cell is in general not bounded (sometimes equal to plus infinity) when the size of the cell goes to infinity. For example, in the pure quantum gravity evoked below, the energy is equal to the curvature minus the local volume which is completely unbounded, when the cells become large. So, it is a really progress to have relaxed this quasilocality assumption.

The proof of Theorem 1 is general and can be applied to many cases of random interacting tessellations. In Sect. 4, we extend our result to the case of geometric hardcore interactions. Indeed, we use our techniques to obtain interesting results in stochastic geometry. Let us suppose that we want to build some random tessellations whose cells are neither too small nor too large. The energy of a triangle is equal to infinity if this triangle is small or large. For example, we put $V(X)$ in (1) equals to plus infinity if the radius of the circumscribed ball of X is smaller than r or bigger than R, where r and R are two positive fixed constants. The Gibbs Delaunay tessellations that we construct in Theorem 2 are therefore more regular than the reference measure P since with probability one there is no small and large cell. Thanks to the DLR equations, we can say that these Gibbs Delaunay tessellations are the most natural random Delaunay tessellations without small and large cells. Finally, let us remark that this kind of hardcore interaction is not authorized in the classical point of view because it is not hereditary. Indeed, let us imagine a tessellation containing a large triangle (the energy is equal to infinity) and let us add a point inside this triangle. So, that it is possible for the energy to become finite. This kind of situation is completely forbidden in the classical case (see Assumption I4 in [17]). Our methods allow us to deal with the non hereditary case.

2 Definitions and Notations

2.1 State Spaces

If $\mathbb{X}$ denotes a Polish space endowed with the Borel σ-algebra $\sigma(\mathbb{X})$, then $\mathcal{B}(\mathbb{X})$ is the set of bounded sets of $\mathbb{X}$. $\mathcal{M}(\mathbb{X})$ is the set of the integer-valued measures Γ on $\mathbb{X}$ such that, for every $\Lambda \in \mathcal{B}(\mathbb{X})$, $\Gamma(\Lambda) \in \mathbb{N}$. $\mathcal{M}(\mathbb{X})$ is endowed with the σ-algebra $\sigma(\mathcal{M}(\mathbb{X}))$ generated by the sets $\{\Gamma \in \mathcal{M}(\mathbb{X}), \Gamma(\Lambda) = n\}$, $n \in \mathbb{N}^*$, $\Lambda \in \mathcal{B}(\mathbb{X})$. Any measure $\Gamma \in \mathcal{M}(\mathbb{X})$ has the following representation

$$\Gamma = \sum_{i \in \mathcal{I}} \delta_{X_i},$$

where $\mathcal{I}$ is a subset of $\mathbb{N}$, $(X_i)_{i\in\mathcal{I}}$ are elements of $\mathbb{X}$ and δ_X is the Dirac measure at X. We write $X \in \Gamma$ if $\Gamma(\{X\}) > 0$. Γ is said simple if for every $X \in \Gamma$, $\Gamma(\{X\}) = 1$.

In our case, $\mathbb{X}$ sometimes denotes $\mathbb{R}^2$ endowed with the Euclidean metric, and sometimes the space $\mathcal{E}$ defined below. So, we note $\mathcal{M}(\mathbb{R}^2)$ the space of integer-valued measures in $\mathbb{R}^2$ and γ denotes a typical element in $\mathcal{M}(\mathbb{R}^2)$. We are interested in a much smaller space than $\mathcal{M}(\mathbb{R}^2)$ containing only the integer-valued measures associated to the Delaunay tessellation. More precisely, we note

$$\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2) = \left\{ \gamma \in \mathcal{M}(\mathbb{R}^2) \text{ such that } \begin{array}{l} \text{– four points of } \gamma \text{ are not on a same circle} \\ \text{– any half plane in } \mathbb{R}^2 \text{ contains some points of } \gamma \end{array} \right\}.$$

We note $\mathcal{E}$ the space

$$\mathcal{E} = \left\{ A \subset \mathbb{R}^2 \text{ such that } \begin{array}{l} – 1 \le \mathrm{Card}(A) \le 3 \\ \text{– A is affinely independent} \end{array} \right\}.$$

Three points are affinely independent if they are not on a same line. In the other cases, it is always true. We note $\mathcal{E}^{(1)} := \{A \in \mathcal{E}, \mathrm{card}(A) = 1\}$ the space of vertices in $\mathbb{R}^2$, $\mathcal{E}^{(2)} := \{A \in \mathcal{E}, \mathrm{card}(A) = 2\}$ the space of edges in $\mathbb{R}^2$ and $\mathcal{E}^{(3)} := \{A \in \mathcal{E}, \mathrm{card}(A) = 3\}$ the space of triangles in $\mathbb{R}^2$. The space $\mathcal{E}$ is endowed with the distance $d_{\mathcal{E}}$ defined below. For $X, Y \in \mathcal{E}$ we note $X \subset Y$ if X is a subset of Y.

For $X, Y \in \mathcal{E}$, we define $N(X) = \mathrm{Card}(X)$ and

$$d_{\mathcal{E}}(X, Y) = \begin{cases} +\infty & \text{if } N(X) \neq N(Y), \\ \max(\sup_{x\in X} \inf_{y\in Y} |x - y|, \sup_{y\in Y} \inf_{x\in X} |x - y|) & \text{if } N(X) = N(Y) \end{cases} \quad (2)$$

where $|.|$ denotes the Euclidean norm in $\mathbb{R}^2$. It is easy to verify that $d_{\mathcal{E}}$ is an distance on $\mathcal{E}$. $(\mathcal{E}, d_{\mathcal{E}})$ is a separable metric space but it is not complete. Consequently, we note $\bar{\mathcal{E}}$ a completion of $\mathcal{E}$ for the metric $d_{\mathcal{E}}$ and $d_{\bar{\mathcal{E}}}$ the extended metric on $\bar{\mathcal{E}}$. We define $\mathcal{M}(\mathcal{E})$ the space of integer-valued measures in $\mathcal{E}$ and Γ denotes a typical element in $\mathcal{M}(\mathcal{E})$. For $\Gamma \in \mathcal{M}(\mathcal{E})$ and $k = 1, 2$ or 3 we note

$$\Gamma^{(k)} = \sum_{\substack{X \in \Gamma \\ N(X)=k}} \delta_X.$$

Let us now define the space of the Delaunay tessellations. For every $X \in \mathcal{E}^{(3)}$, we note $B(X)$ the open circumscribed ball of X and $\bar{B}(X)$ the closure of $B(X)$.

$\Gamma \in \mathcal{M}(\mathcal{E})$ is a **Delaunay tessellation** if

(i) $\Gamma^{(1)}$ is in $\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)$
(ii) for every $X \in \Gamma$ for every $Y \subset X$ then $Y \in \Gamma$
(iii) for every $Y \in \Gamma^{(1)} \cup \Gamma^{(2)}$, there exist two elements $X, X' \in \Gamma^{(3)}$ such that $Y \subset X$, $Y \subset X'$
(iv) for every $X \in \Gamma^{(3)}$, $\Gamma^{(1)}(B(X)) = 0$.

We note $\mathcal{M}_{\mathcal{D}}(\mathcal{E})$ the space of the Delaunay tessellations.

The conditions (ii), (iii) give to Γ the structure of simplicial complex (for more details, see [20]). The property (iv) is a typical property for the Delaunay tessellation (see [12]).

Let us remark that a Delaunay tessellation is uniquely and completely determined by $\Gamma^{(1)}$ (see [12]). So for every $\gamma \in \mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)$ we denote by $\bar{\gamma}$ the unique Delaunay tessellation Γ in $\mathcal{M}_{\mathcal{D}}(\mathcal{E})$ such that $\Gamma^{(1)} = \gamma$.

For every $X \in \mathcal{E}$, we note $\langle X \rangle$ the convex hull of X in $\mathbb{R}^2$ and for every $\Gamma \in \mathcal{M}(\mathcal{E})$ we note $\langle \Gamma \rangle = \bigcup_{X \in \Gamma} \langle X \rangle$. Let us remark that for every $\Gamma \in \mathcal{M}_{\mathcal{D}}(\mathcal{E})$, $\langle \Gamma \rangle = \mathbb{R}^2$. Let Γ, Γ' are in $\mathcal{M}(\mathcal{E})$, we note $\Gamma' \prec \Gamma$ if the measure Γ' is absolutely continuous with respect to Γ. In fact, $\Gamma' \prec \Gamma$ if for every X in Γ' then X is in Γ.

2.2 Interaction

In this section, we define the energy of finite connected configurations. At this stage, we do not give the necessary assumptions in the following theorems to prove the existence of Gibbs Delaunay tessellations. Until the end of the paper, the fixed integer p is the maximum number of cells for a typical interacting configuration. More precisely, we note

$$\mathcal{S}_p(\mathcal{E}) = \left\{ \Gamma \in \mathcal{M}(\mathcal{E}) \text{ such that } \begin{array}{l} -\ \Gamma(\mathcal{E}) \le p \\ -\ \langle \Gamma \rangle := \bigcup_{X \in \Gamma} \langle X \rangle \text{ is a connected set} \end{array} \right\}.$$

Let us remark that an element Γ in $\mathcal{S}_p(\mathcal{E})$ does not satisfy in general the assumptions (i), (ii), (iii), (iv). An interaction potential V is a measurable function

$$V : \mathcal{S}_p(\mathcal{E}) \longrightarrow \mathbb{R} \cup \{+\infty\}.$$

Now, we can define the local energy of an infinite Delaunay tessellation. Let Λ be a bounded set in $\mathcal{B}(\mathbb{R}^2)$ and Γ a Delaunay tessellation in $\mathcal{M}(\mathcal{E})$. We define the energy of Γ inside Λ with the following definition

$$E_\Lambda(\Gamma) = \sum_{\substack{\Gamma' \prec \Gamma \\ \Gamma' \in \mathcal{S}_p(\mathcal{E}) \\ \langle \Gamma' \rangle \cap \Lambda \neq \emptyset}} V(\Gamma'). \tag{3}$$

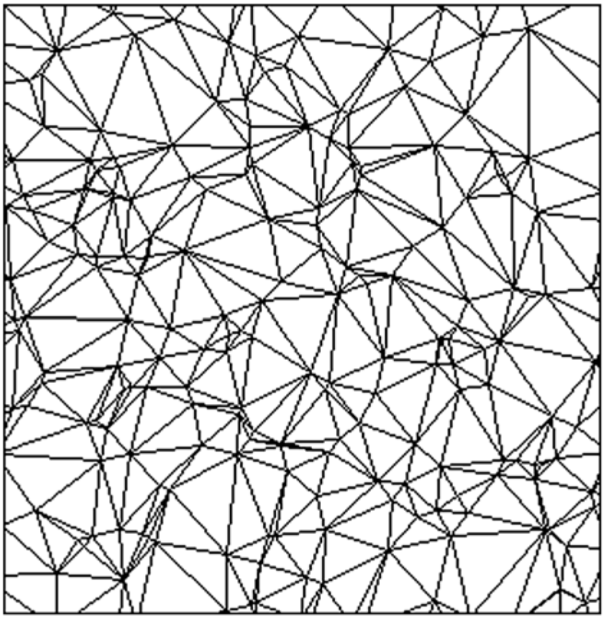

Fig. 1 An example of a Delaunay tessellation

It can be seen that the sum above is finite thanks to the locality of the Delaunay tessellation. Since Γ is determined by γ, we can define the energy of γ inside Λ by the simple formula $E_\Lambda(\gamma) := E_\Lambda(\bar{\gamma})$. We use the same notation for the energy of Γ in Λ and the energy of γ in Λ. We note $\mathcal{M}_\mathcal{D}^\infty(\mathbb{R}^2)$ (respectively $\mathcal{M}_\mathcal{D}^\infty(\mathcal{E})$) the subset of $\mathcal{M}_\mathcal{D}(\mathbb{R}^2)$ (respectively $\mathcal{M}_\mathcal{D}(\mathcal{E})$) such that the configurations γ (respectively Γ) have a finite local energy $E_\Lambda(\gamma) < +\infty$ (respectively $E_\Lambda(\Gamma) < +\infty$) for all $\Lambda \in \mathcal{B}(\mathbb{R}^2)$.

This definition of the energy is similar to the one given in [20].

2.3 The Reference Measure and the Local Specifications

We note λ the Lebesgue measure on $\mathbb{R}^2$, π denotes the Poisson Process on $\mathbb{R}^2$ with intensity λ. It is a probability measure on $\mathcal{M}(\mathbb{R}^2)$. For every $\Lambda \in \mathcal{B}(\mathbb{R}^2)$, π_Λ denotes the Poisson process on Λ with intensity $\mathbb{1}_\Lambda \lambda$. We note $\bar{\pi}$ the natural Poisson Delaunay tessellation which is the image of π to $\mathcal{M}_\mathcal{D}(\mathcal{E})$ by the application $\gamma \mapsto \bar{\gamma}$. $\bar{\pi}$ is a Probability measure on $\mathcal{M}_\mathcal{D}(\mathcal{E})$ and is the reference process for the Gibbs tessellations. It can be seen as the random Delaunay tessellation without interaction. In fact, we want to construct Gibbs Delaunay tessellations which are locally absolutely continuous with respect to $\bar{\pi}$. The local density is given by local specifications that we are currently defining.

First, let us explain how the reference measure $\bar{\pi}$ satisfies the local conditioning. In the classical continuous case, π is the reference measure and consequently local conditioning is trivial. For every $\Lambda \in \mathcal{B}(\mathbb{R}^2)$, the process γ under π has two independent parts γ_Λ (the projection of γ inside Λ) and γ_{Λ^c} (the projection of γ outside Λ). In our case, it is more complicated, because the inside and outside of under $\bar{\pi}$ are not independent. Moreover what exactly are the inside and outside of Λ? There are many different ways of defining them. In [5–7], only the points of $\Gamma^{(1)}$ are considered. So, the inside and outside are clearly and easily defined but they lose the structure of the tessellation near the boundary of Λ because a triangle in Λ^c which has a circumscribing ball intersecting Λ disappears if you add a point

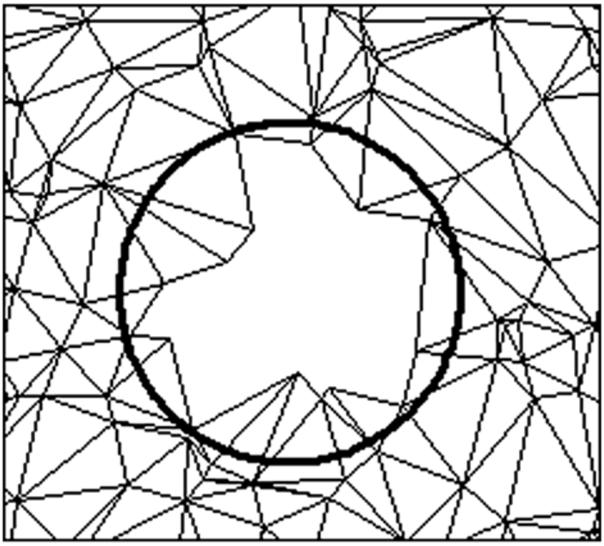

Fig. 2 An example of configuration Γ_{Λ^c}

at $\Gamma^{(1)}$ in Λ and in this ball. Consequently, the tessellation is not stable near the boundary and there is an influence of points inside of Λ on the tessellation outside of Λ.

In our case, we prefer to keep the structure of the Delaunay tessellation completely outside of Λ. More precisely, we have the following definition.

Let Λ be in $\mathcal{B}(\mathbb{R}^2)$ and $\Gamma \in \mathcal{M}_{\mathcal{D}}(\mathcal{E})$, we note Γ_{Λ^c} the process

$$\Gamma_{\Lambda^c} = \sum_{\substack{X \in \Gamma \\ \exists Y \in \Gamma^{(3)},\ X \subset Y \\ \langle Y \rangle \cap \Lambda^c \neq \emptyset}} \delta_X. \tag{4}$$

In fact, Γ_{Λ^c} corresponds to the part of the Delaunay tessellation Γ which intersects the outside of Λ. Let us remark that every $X \in \Gamma_{\Lambda^c}$ does not necessary satisfy $\langle X \rangle \cap \Lambda^c \neq \emptyset$ but it is contained in a triangle Y which satisfies it. We have in Fig. 2 an example of configuration Γ_{Λ^c}.

In the following proposition we identify the law of Γ under $\bar{\pi}$ given Γ_{Λ^c}. Let us define before the set $\mathcal{S}_\Lambda(\Gamma)$ in $\mathbb{R}^2$

$$\mathcal{S}_\Lambda(\Gamma) = \mathbb{R}^2 \setminus \Bigg(\bigcup_{X \in \Gamma^{(3)}_{\Lambda^c}} \bar{B}(X) \Bigg) \tag{5}$$

and the function Υ

$$\begin{aligned} \Upsilon \colon \mathcal{M}_{\mathcal{D}}(\mathcal{E}) \times \mathcal{M}(\mathbb{R}^2) \times \mathcal{B}(\mathbb{R}^2) &\longrightarrow \mathcal{M}_{\mathcal{D}}(\mathcal{E}), \\ (\Gamma, \gamma, \Lambda) &\longmapsto \overline{(\Gamma^{(1)}_{\mathcal{S}_\Lambda(\Gamma)^c} + \gamma_{\mathcal{S}_\Lambda(\Gamma)})}. \end{aligned}$$

With this definition, $\Upsilon(\Gamma, \gamma, \Lambda)$ is not correctly defined if $\Gamma^{(1)}_{\mathcal{S}_\Lambda(\Gamma)^c} + \gamma_{\mathcal{S}_\Lambda(\Gamma)}$ is not in $\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)$ (it is possible if $\Gamma^{(1)}_{\mathcal{S}_\Lambda(\Gamma)^c} + \gamma_{\mathcal{S}_\Lambda(\Gamma)}$ has four points on the same circle). In this case, we put $\Upsilon(\Gamma, \gamma, \Lambda) = \Gamma$.

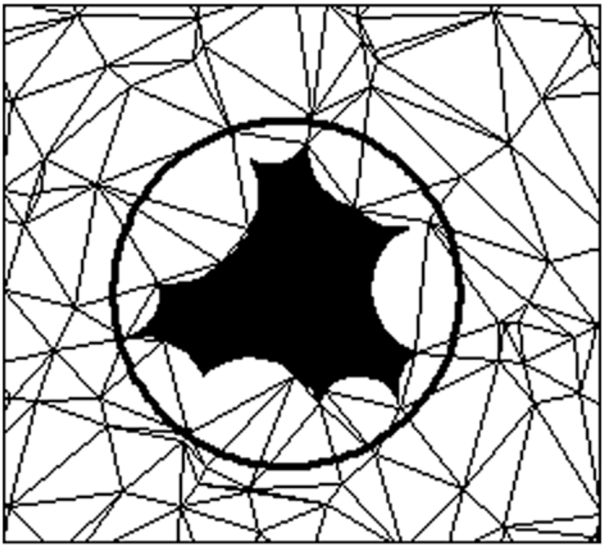

Fig. 3 An example of set $\mathcal{S}_\Lambda(\Gamma)$ represented as the dark area

The function Υ extends the partial Delaunay tessellation Γ_{Λ^c} inserting the points of γ inside $\mathcal{S}_\Lambda(\Gamma)$ and completing the Delaunay structure.

Proposition 1 *Let Λ be a bounded set in $\mathbb{R}^2$. Then, for every bounded measurable function f we have*

$$\int_{\mathcal{M}_\mathcal{D}(\mathcal{E})} f(\Gamma)\bar{\pi}(d\Gamma) = \int_{\mathcal{M}_\mathcal{D}(\mathcal{E})}\int_{\mathcal{M}(\mathbb{R}^2)} f(\Upsilon(\Gamma,\gamma,\Lambda))\pi_{\mathcal{S}_\Lambda(\Gamma)}(d\gamma)\bar{\pi}(d\Gamma). \tag{6}$$

Proof We have for all Λ in $\mathbb{R}^2$ and all f, g measurable bounded functions on $\mathcal{M}(\mathcal{E})$

$$\begin{aligned}\int_{\mathcal{M}_\mathcal{D}(\mathcal{E})} f(\Gamma)\bar{\pi}(d\Gamma) &= \int_{\mathcal{M}(\mathbb{R}^2)} f(\bar{\gamma})\pi(d\gamma)\\ &= \int_{\mathcal{M}(\mathbb{R}^2)}\int_{\mathcal{M}(\mathbb{R}^2)} f(\bar{\gamma})\pi(d\gamma|\bar{\gamma}_{\Lambda^c} = \bar{\gamma}'_{\Lambda^c})\pi(d\gamma').\end{aligned}$$

Since $\Upsilon(\Gamma,\gamma,\Lambda)$ depends only on Γ_{Λ^c} and $\gamma_{\mathcal{S}_\Lambda(\Gamma)}$, we have

$$\begin{aligned}&\int_{\mathcal{M}_\mathcal{D}(\mathcal{E})} g(\Gamma_{\Lambda^c})f(\Gamma)\bar{\pi}(d\Gamma)\\ &\quad= \int_{\mathcal{M}(\mathbb{R}^2)}\int_{\mathcal{M}(\mathbb{R}^2)} f(\Upsilon(\bar{\gamma}',\gamma,\Lambda))\pi(d\gamma|\bar{\gamma}_{\Lambda^c} = \bar{\gamma}'_{\Lambda^c})\pi(d\gamma')\\ &\quad= \int_{\mathcal{M}_\mathcal{D}(\mathcal{E})}\int_{\mathcal{M}(\mathbb{R}^2)} f(\Upsilon(\Gamma,\gamma,\Lambda))\pi(d\gamma|\bar{\gamma}_{\Lambda^c} = \Gamma_{\Lambda^c})\bar{\pi}(d\Gamma)\\ &\quad= \int_{\mathcal{M}_\mathcal{D}(\mathcal{E})}\int_{\mathcal{M}(\mathbb{R}^2)} f(\Upsilon(\Gamma,\gamma,\Lambda))\pi(d\gamma|\gamma_{\mathcal{S}_\Lambda(\Gamma)^c} = \Gamma^{(1)}_{\mathcal{S}_\Lambda(\Gamma)^c})\bar{\pi}(d\Gamma)\\ &\quad= \int_{\mathcal{M}_\mathcal{D}(\mathcal{E})}\int_{\mathcal{M}(\mathbb{R}^2)} f(\Upsilon(\Gamma,\gamma,\Lambda))\pi_{\mathcal{S}_\Lambda(\Gamma)}(d\gamma)\bar{\pi}(d\Gamma).\end{aligned}$$

The proposition is proved. □

This proposition gives the conditional law under $\bar{\pi}$ of Γ given Γ_{Λ^c}. It is the first step in defining the D.L.R. equations. This description of $\bar{\pi}$ given Γ_{Λ^c} gives us a natural description of the local law of $\bar{\pi}$ and we will base our Gibbs structure on this result. It is not really new because it is in the spirit of the geometric Gibbsian structure for the Arak processes (see [1–3]) and it is suggested by Zessin in [20] to obtain a purely geometric description of kernels for Gibbs simplicial complexes. Let us now give the local specifications with interaction. For every $\Lambda \in \mathcal{B}(\mathbb{R}^2)$, we define the kernel Ξ_Λ on $\mathcal{M}_{\mathcal{D}}^\infty(\mathcal{E}) \times \mathcal{P}(\mathcal{M}_{\mathcal{D}}^\infty(\mathcal{E}))$. So, for every $\Gamma \in \mathcal{M}_{\mathcal{D}}^\infty(\mathcal{E})$ and every bounded continuous function f from $\mathcal{M}_{\mathcal{D}}^\infty(\mathcal{E})$ to $\mathbb{R}$ we define

$$\Xi_\Lambda(\Gamma, f) = \int_{\mathcal{M}(\mathbb{R}^2)} f(\Upsilon(\Gamma,\gamma,\Lambda)) \frac{1}{Z_\Lambda(\Gamma)} e^{-E_\Lambda(\Upsilon(\Gamma,\gamma,\Lambda))} \pi_{\mathcal{S}_\Lambda(\Gamma)}(d\gamma), \tag{7}$$

where $Z_\Lambda(\Gamma)$ is the normalization constant

$$Z_\Lambda(\Gamma) = \int_{\mathcal{M}(\mathbb{R}^2)} e^{-E_\Lambda(\Upsilon(\Gamma,\gamma,\Lambda))} \pi_{\mathcal{S}_\Lambda(\Gamma)}(d\gamma).$$

We have to justify that the constant $Z_\Lambda(\Gamma)$ is finite and not null for all $\Gamma \in \mathcal{M}_{\mathcal{D}}^\infty(\mathcal{E})$. Since that depends on the interaction V, this will be proved later when we introduce precisely the assumptions on V. In fact that comes from the stability of V and a hardcore property. We have to extend the definition of $\Xi_\Lambda(\Gamma, .)$ in the case where $\mathcal{S}_\Lambda(\Gamma) = \emptyset$. In this case, we put $\Xi_\Lambda(\Gamma, f) = f(\Gamma)$ and the probability measure $\Xi_\Lambda(\Gamma, .)$ is in fact δ_Γ.

Now, the family of kernel $(\Xi_\Lambda)_{\Lambda \in \mathcal{B}(\mathbb{R}^2)}$ is a specification if the kernels are **compatible**. It means that for every bounded measurable function f on $\mathcal{M}_{\mathcal{D}}(\mathcal{E})$, every Γ in $\mathcal{M}_{\mathcal{D}}^\infty(\mathcal{E})$ and every Λ, Λ' in $\mathcal{B}(\mathbb{R}^2)$ such that $\Lambda \subset \Lambda'$ then

$$\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} \int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} f(\Gamma'') \Xi_\Lambda(\Gamma', d\Gamma'') \Xi_{\Lambda'}(\Gamma, d\Gamma') = \int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} f(\Gamma') \Xi_{\Lambda'}(\Gamma, d\Gamma'). \tag{8}$$

In general this fact is obvious in statistical mechanics; that comes from the additivity of the energy. But in our case the space state and interaction are a little different, we prefer to give a brief validation of this result.

Proposition 2 *The kernels $(\Xi_\Lambda)_{\Lambda \in \mathcal{B}(\mathbb{R}^2)}$ are compatible.*

Proof Let $\Lambda, \Lambda' \in \mathcal{B}(\mathbb{R}^2)$, $\Gamma \in \mathcal{M}_{\mathcal{D}}^\infty(\mathcal{E})$ and f a bounded measurable function on $\mathcal{M}_{\mathcal{D}}(\mathcal{E})$. We suppose $\Lambda \subset \Lambda'$. So, we have

$$\begin{aligned}
\Xi_{\Lambda'}(\Gamma, f) &= \int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} \left[\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} f(\Gamma'') \Xi_{\Lambda'}(\Gamma, d\Gamma'' | \Gamma''_{\Lambda^c} = \Gamma'_{\Lambda^c}) \right] \Xi_{\Lambda'}(\Gamma, d\Gamma') \\
&= \int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} \left[\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} f(\Gamma'') \left(\frac{1}{Z_{\Lambda'}(\Gamma)} e^{-E_\Lambda(\Gamma'') - (E_{\Lambda'}(\Gamma'') - E_\Lambda(\Gamma''))} \Xi^0_{\Lambda'} \right) \right. \\
&\qquad \left. \times (\Gamma, d\Gamma'' | \Gamma''_{\Lambda^c} = \Gamma'_{\Lambda^c}) \right] \Xi_{\Lambda'}(\Gamma, d\Gamma'),
\end{aligned}$$

where $(\Xi^0_\Lambda)_{\Lambda \in \mathcal{B}(\mathbb{R}^2)}$ are the kernels without interaction ($V = 0$) and with the convention $E_{\Lambda'}(\Gamma'') - E_\Lambda(\Gamma'') = 0$ if $E_{\Lambda'}(\Gamma'') = +\infty$ and $E_\Lambda(\Gamma'') = +\infty$. Thanks to the Proposition 1

$$\Xi^0_{\Lambda'}(\Gamma, d\Gamma'' | \Gamma''_{\Lambda^c} = \Gamma'_{\Lambda^c}) = \Xi^0_\Lambda(\Gamma', d\Gamma'').$$

So, we have

$$\Xi_{\Lambda'}(\Gamma,f)=\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})}\left[\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} f(\Gamma'')\frac{1}{Z_{\Lambda'}(\Gamma)}e^{-E_\Lambda(\Gamma'')-(E_{\Lambda'}(\Gamma'')-E_\Lambda(\Gamma''))}\Xi^0_\Lambda(\Gamma',d\Gamma'')\right]$$
$$\times\left[\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})}\frac{1}{Z_{\Lambda'}(\Gamma)}e^{-E_\Lambda(\Gamma'')-(E_{\Lambda'}(\Gamma'')-E_\Lambda(\Gamma''))}\Xi^0_\Lambda(\Gamma',d\Gamma'')\right]^{-1}\Xi_{\Lambda'}(\Gamma,d\Gamma').$$

Since $E_{\Lambda'}(\Gamma)-E_\Lambda(\Gamma)$ and $Z_{\Lambda'}(\Gamma)$ depend only on Γ_{Λ^c}, we have

$$\begin{aligned}\Xi_{\Lambda'}(\Gamma,f)&=\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})}\left[\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} f(\Gamma'')\frac{1}{Z_\Lambda(\Gamma')}e^{-E_\Lambda(\Gamma'')}\Xi^0_\Lambda(\Gamma',d\Gamma'')\right]\Xi_{\Lambda'}(\Gamma,d\Gamma')\\&=\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})}\left[\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} f(\Gamma'')\Xi_\Lambda(\Gamma',d\Gamma'')\right]\Xi_{\Lambda'}(\Gamma,d\Gamma').\end{aligned}$$

The proposition is proved. □

Now we can define the Gibbs Delaunay tessellations.

Definition 1 *A probability measure μ on $\mathcal{M}_{\mathcal{D}}(\mathcal{E})$ is a Gibbs Delaunay tessellation for the interaction potential V if, for every bounded set Λ in $\mathcal{B}(\mathbb{R}^2)$ and every bounded measurable function f from $\mathcal{M}_{\mathcal{D}}(\mathcal{E})$ to $\mathbb{R}$, we have*

$$\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} f(\Gamma)\mu(d\Gamma)=\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})}\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} f(\Gamma')\Xi_\Lambda(\Gamma,d\Gamma')\mu(d\Gamma). \tag{9}$$

The equations (9) are called DLR equations (Dobrushin, Landford, Ruelle), they generalize the equations (6) in the case with interaction. It is equivalent to: for μ almost every Γ, every bounded set Λ in $\mathcal{B}(\mathbb{R}^2)$ and every bounded measurable function f from $\mathcal{M}_{\mathcal{D}}(\mathcal{E})$ to $\mathbb{R}$

$$\mu(f|\Gamma_{\Lambda^c})=\Xi_\Lambda(\Gamma,f).$$

3 Existence of Gibbs Delaunay Tessellations without Hardcore Condition

In this section, our results are a direct extension of the results in [6]. In this paper, the authors prove the existence of Gibbs Delaunay tessellations for an interaction on the finite configurations, which is null if the configuration is included in a triangle having a minimal angle smaller than a fixed angle $\alpha_0>0$. Moreover, they assume the quasilocality of the interaction. It means that the interaction goes to zero when the diameter of the configuration goes to infinity. With these two assumptions, they can use the Preston theorem (see [14]) to prove the existence of Gibbs Delaunay tessellations (see p. 727 in [6] and p. 899 in [7]). In Theorem 1, we generalize this result by relaxing the quasilocality assumption.

Our methods in this paper are completely different. Firstly, the Gibbs Delaunay tessellations concept is different. Nevertheless, in the case without hardcore interactions we can prove that their concept, which is purely a point process concept, and our concept of Gibbs Delaunay tessellations, which is more geometric, are equivalent. It will not be the case in the next section. Secondly, we do not use Preston's Theorems because they require a uniform quasilocality assumption on the kernels. In [6, 7], the quasilocality of the interaction guarantees this assumption. With our methods, the locality of the Delaunay tessellations is sufficient.

Let us define the assumptions on V in this case without hardcore. We suppose until the end of this section that V satisfies the following assumptions.

(TI): The translation invariant condition

$$\forall h \in \mathbb{R}^2,\ \forall \Gamma \in \mathcal{S}_p(\mathcal{E}) \quad V(\tau_h \Gamma) = V(\Gamma),$$

where τ_h is the shift operator for the vector $h \in \mathbb{R}^2$. Now, let us give the assumption which assures that V does not have a hardcore part.

(F): The finite energy condition

$$\forall \Gamma \in \mathcal{S}_p(\mathcal{E}), \quad V(\Gamma) < +\infty.$$

Let X be a triangle in $\mathcal{E}^{(3)}$, we note $\alpha(X)$ the smallest angle in X. Let us now define a condition about the support of V.

(S): The support condition

there exists $\alpha_0 > 0$ such that for every Γ in $\mathcal{S}_p(\mathcal{E})$.

$$V(\Gamma) \neq 0 \Longrightarrow \forall X \in \Gamma\ \exists Y \in \Gamma^{(3)} \text{ such that } X \subset Y \text{ and } \alpha(Y) > \alpha_0.$$

(S) means that V only charges the configurations Γ which do not have too flat triangles. The interaction potential is bounded if

(B): The bounded condition

$\exists M > 0,\ \forall \Gamma \in \mathcal{S}_p(\mathcal{E}),$ then $V(\Gamma) < +\infty \Longrightarrow |V(\Gamma)| \leq M.$

There is a last assumption about the regularity of V.

(R): The regular condition

V is $\overline{\pi}$ a.s. continuous. This means there exists $\Omega \subset \mathcal{M}_{\mathcal{D}}(\mathcal{E})$ with $\overline{\pi}(\Omega) = 1$ such that for every $\Gamma \in \Omega$, every $\Gamma' \prec \Gamma$ in $\mathcal{S}_p(\mathcal{E})$, V is continuous at Γ'.

Now, let us give some remarks and consequences of these assumptions.

(TI) is needed because we study only invariant translation Gibbs Delaunay tessellations. Moreover, we need this invariant translation to construct the Gibbs tessellation and to control the convergence of the finite volume Gibbs tessellations. The conditions (S) and (B) are the same as in [6]. From a theoretical point of view, they are strong but from a practical point of view, they are very weak because we can choose α_0 very small and M very large. The condition (R) is a little technical but it is satisfied for every natural interaction. This condition is not needed in [6].

If the interaction V satisfies (F), we can define the energy of point x in a configuration γ. Let $x \in \mathbb{R}^2$ and $\gamma \in \mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)$ such that $\gamma + \delta_x$ is in $\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)$. Then, we define the energy of the point x in γ by the following formula

$$E(x,\gamma) = \sum_{\substack{\Gamma' \prec \overline{\gamma + \delta_x} \\ \Gamma' \nprec \overline{\gamma}}} V(\Gamma') - \sum_{\substack{\Gamma' \prec \overline{\gamma} \\ \Gamma' \nprec \overline{\gamma + \delta_x}}} V(\Gamma'). \tag{10}$$

Let us remark that there is only a finite number of summands in each of the sums above. So, thanks to this definition we have

$$E_\Lambda(\gamma) = E_\Lambda(\gamma_{\Lambda^c}) + \sum_{i=1}^{k} E(x_i, \gamma_{\Lambda^c} + \delta_{x_1} + \cdots + \delta_{x_{i-1}}), \tag{11}$$

where $\gamma_\Lambda = \delta_{x_1} + \cdots + \delta_{x_k}$.

Let us remark that the definition of kernels $(\Xi_\Lambda)_{\Lambda\subset\mathbb{R}^2}$ in (7) would be exactly the same (in the case where V satisfies (F)) if we substituted $E_\Lambda(\gamma)$ with the cumulated sum of the energy (i.e. the second part in the right term in equality (11)) because $E_\Lambda(\gamma_{\Lambda^c})$ is a constant. This is the reason for which the techniques used in this section are based partially on classical tools for the point processes. For instead, we will use the reduced Campbell measure on $\mathcal{M}(\mathbb{R}^2)$ that we remind below.

Let P be a probability measure on $\mathcal{M}(\mathbb{R}^2)$, we define the reduced Campbell measure $C^!_P$ on $\mathcal{M}(\mathbb{R}^2)\times\mathbb{R}^2$ by

$$\mathcal{C}^!_P(f) = \iint f(x, \gamma - \delta_x)\gamma(dx)P(d\gamma),$$

where f is a bounded measurable function from $\mathcal{M}(\mathbb{R}^2)\times\mathbb{R}^2$ to $\mathbb{R}$. Now let us give the first Theorem.

Theorem 1 *For any* $\alpha_0 > 0$, *there exists a Gibbs Delaunay tessellation for a potential* V *which satisfies* (TI), (F), (S), (B) *and* (R).

Proof First of all, we have to prove that the kernels (Ξ_Λ) (see (7)) are well defined. Let us show that $0 < Z_L(\Gamma) < +\infty$ for every $\Gamma \in \mathcal{M}^\infty_{\mathcal{D}}(\mathcal{E})$. The definition of $Z_\Lambda(\Gamma)$ is (we suppose that $\mathcal{S}_\Lambda(\Gamma) \neq \emptyset$)

$$Z_\Lambda(\Gamma) = \int_{\mathcal{M}(\mathbb{R}^2)} e^{-E_\Lambda(\Upsilon(\Gamma,\gamma,\Lambda))}\pi_{\mathcal{S}_\Lambda(\Gamma)}(d\gamma).$$

Since there is no hardcore, it is obvious that $Z_\Lambda(\Gamma) > 0$. Now, we are going to prove the stability.

Let x be in $\Gamma^{(1)}$ and $\Gamma' \prec \Gamma$ be in $\mathcal{S}_p(\mathcal{E})$ such that $V(\Gamma') \neq 0$. Thanks to the assumption (S), Γ' is included, at the maximum, in p connected triangles each having angles bigger than α_0. The number of such p connected triangles configurations in Γ with large angles containing x is bounded by $C := (p\frac{2\pi}{\alpha_0})^p$. This controls the first sum of the energy $E(x, \Gamma^{(1)})$ defined in (10) since $|V(.)| \leq M$ (M coming from the assumption (B)). For the second part, let us point out that the number of triangles $X \in \overline{\Gamma^{(1)} - \delta_x}$ such that $\alpha(X) > \alpha_0$ and $X \notin \Gamma$ is bounded by $\frac{2\pi}{\alpha_0}$ (see Proposition 3 in [7]). So the number of configurations $\Gamma' \in \mathcal{S}_p(\mathcal{E})$ included in $\overline{\Gamma^{(1)} - \delta_x}$ and not included in Γ such that $V(\Gamma) \neq 0$ is also bounded by the constant C. We deduce that $|E(x,\gamma)|$ is bounded by $2MC$ and the interaction V is stable. $Z_\Lambda(\Gamma)$ is finite and nonnull.

Now, the proof of the theorem can be sketched into two parts. The first is to find and construct a probability measure μ in $\mathcal{M}_{\mathcal{D}}(\mathcal{E})$ that will be a good candidate to become a Gibbs measure. So we define the bounded periodic Gibbs tessellation μ_n associated to the potential V and prove the convergence of these measures to a measure μ. Afterwards, in a second part we show that μ satisfies the DLR equations characterizing the Gibbs structure. In this second part, we have to prove essentially that the kernels $(\Xi_\Lambda)_{\Lambda\in\mathcal{B}(\mathbb{R}^2)}$ are continuous for μ almost every Γ with respect to the topology that we use.

Let us now define the stationary periodic Gibbs tessellation.

Let Λ_n be the bounded set $[-n,n]^2 \subset \mathbb{R}^2$. We define the stationary periodic Gibbs tessellation μ_n for every $n \in \mathbb{N}^*$ on Λ_n. In general, the periodic Gibbs measure is defined on a torus. We prefer to define it on the space $\mathcal{M}_{\mathcal{D}}(\mathcal{E})$ with periodic configurations. Let us now introduce some notations.

For every $x \in \mathbb{R}^2$, we note $p_n(x)$ the configuration $\sum_{h\in 2n\mathbb{Z}^d} \delta_{x+h}$. For every $\gamma \in \mathcal{M}(\Lambda_n)$ we note $p_n(\gamma)$ the periodic configuration $p_n(\gamma) = \sum_{x\in\gamma} p_n(x)$. An element Γ in $\mathcal{M}_{\mathcal{D}}(E)$ is called n-periodic if, for every $h \in 2n\mathbb{Z}^d$, $\tau_h(\Gamma) = \Gamma$. We clearly see that for every $\gamma \in \mathcal{M}(\Lambda_n)$, $\overline{p_n(\gamma)}$ is n-periodic.

Now, for every $\Gamma \in \mathcal{M}_{\mathcal{D}}(\mathcal{E})$, we define an equivalence relation on $\{\Gamma' \in \mathcal{S}_p(\mathcal{E})$ such that $\Gamma' \prec \Gamma\}$ by : Γ' and Γ'' are equivalent if there exists $h \in 2n\mathbb{Z}^d$ such that $\tau_h(\Gamma') = \Gamma''$. We note $\mathcal{T}_n(\Gamma)$ the set of equivalence classes, an element in $\mathcal{T}_n(\Gamma)$ is noted $\tilde{\Gamma}'$ with $\Gamma' \prec \Gamma$.

For two different elements $\Gamma' \prec \Gamma$ and $\Gamma'' \prec \Gamma$ in $\mathcal{S}_p(\mathcal{E})$ such that $\tilde{\Gamma}' = \tilde{\Gamma}''$, we have $V(\Gamma') = V(\Gamma'')$ since V satisfies (TI). Consequently we can define the periodic energy of a *n-periodic* configuration $\Gamma \in \mathcal{M}_{\mathcal{D}}(\mathcal{E})$ in a volume $\Lambda \subset \Lambda_n$ by

$$E^n_\Lambda(\Gamma) = \sum_{\substack{\tilde{\Gamma}'\in\mathcal{T}_n(\Gamma)\\ \langle\Gamma'\rangle\cap\Lambda\neq\emptyset}} V(\Gamma'). \tag{12}$$

Let us now define the periodic Gibbs Delaunay tessellation μ_n: for every positive bounded function f from $\mathcal{M}_{\mathcal{D}}(\mathcal{E})$ to $\mathbb{R}$ we have

$$\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} f(\Gamma)\mu_n(d\Gamma) = \int_{\mathcal{M}(\Lambda_n)} \frac{1}{Z_n} f(\overline{p_n(\gamma)})e^{-E^n_{\Lambda_n}(\overline{p_n(\gamma)})}\pi_{\Lambda_n}(d\gamma), \tag{13}$$

with $Z_n = \int_{\mathcal{M}(\Lambda_n)} e^{-E^n_{\Lambda_n}(\overline{p_n(\gamma)})}\pi_{\Lambda_n}(d\gamma)$. If $\gamma = 0$ or $\gamma = \delta_x$ then we put $\overline{p_n(\gamma)} = 0$ and $E^n_{\Lambda_n}(\overline{p_n(\gamma)}) = 0$. Let us remark that Z_n is finite for all $n \in \mathbb{N}^*$ because V is stable; see the calculus above.

First step, convergence of (μ_n):

Let us look at precisely the convergence concept. The converge of the probability measures μ_n is for the weak topology on $\mathcal{P}(\mathcal{M}(\mathcal{E}))$ where the topology on $\mathcal{M}(\mathcal{E})$ is induced by the metric $d_{\mathcal{M}(\mathcal{E})}$. $d_{\mathcal{M}(\mathcal{E})}$ is the associated metric for the vague topology on $\mathcal{M}(\mathcal{E})$, $\mathcal{E}$ being endowed by the metric $d_{\mathcal{E}}$ defined in (2). See for example [8] p. 607 for an appendix about the topology of the convergence of measures and [11] p. 108 for the convergence of integer-valued measures.

Let us begin by analyzing the metric $d_{\mathcal{E}}$ and remembering that the set $\mathcal{E}$ is not a complete space. So, for the moment, we have to define the convergence in $\bar{\mathcal{E}}$ for which we have the extended metric $d_{\bar{\mathcal{E}}}$. $d_{\bar{\mathcal{E}}}(X)$ is defined by the limit of $d_{\mathcal{E}}(X_n)$, when n goes to infinity, and for any set (X_n) which tends to X. So, a set $(X_n)_{n\in\mathbb{N}^*} \in \mathcal{E}^{\mathbb{N}^*}$ converges to an element $X \in \bar{\mathcal{E}}$ if, and only if, there exists n_0 in $\mathbb{N}^*$ such that $\forall n \geq n_0,\ N(X_n) = N(X)$ and we can write X_n like $\{x_n\}$ or $\{x_n, y_n\}$ or $\{x_n, y_n, z_n\}$ (idem for X, $\{x\}$ or $\{x, y\}$ or $\{x, y, z\}$) with $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} y_n = y$.

Next, let us look at the metric $d_{\mathcal{M}(\bar{\mathcal{E}})}$ (for a precise definition, we can see [11] pp. 108–109). A set of measures $(\Gamma_n) \in \mathcal{M}(\mathcal{E})^{\mathbb{N}^*}$ converges to $\Gamma \in \mathcal{M}(\bar{\mathcal{E}})$ if, and only if, for every open bounded set $\Lambda \subset \bar{\mathcal{E}}$, there exists $\varepsilon > 0$ and $n_0 \in \mathbb{N}^*$ such that for all $n \geq n_0$ $\Gamma_n(\Lambda^\varepsilon) = \Gamma(\Lambda^\varepsilon) = \Gamma(\Lambda)$ with

$$\Lambda^\varepsilon = \{X \in \bar{\mathcal{E}} \text{ such that } B(X, \varepsilon) \subset \Lambda\}.$$

Moreover, we can write $(\Gamma_n)_{\Lambda^\varepsilon} = \sum_{k=1}^{\Gamma(\Lambda)} \delta_{X^n_k}$, $\Gamma_{\Lambda^\varepsilon} = \sum_{k=1}^{\Gamma(\Lambda)} \delta_{X_k}$, and we have $\lim_{n\to\infty} X^n_k = X_k$ for every $1 \leq k \leq \Gamma(\Lambda)$. The set Λ^ε is introduced to control the boundary effects which can appear if the configurations Γ_n have points near the boundary of Λ. The convergence of Γ_n to Γ for the metric $d_{\mathcal{M}(\bar{\mathcal{E}})}$ is equivalent to the convergence for the vague topology on

$\mathcal{M}(\bar{\mathcal{E}})$. It means that for every bounded nonnegative continuous function f on $\bar{\mathcal{E}}$, which is identically null outside of suitable bounded set of $\bar{\mathcal{E}}$, we have

$$\lim_{n\to\infty}\int_{\bar{\mathcal{E}}} f(X)\Gamma_n(dX) = \int_{\bar{\mathcal{E}}} f(X)\Gamma(dX).$$

Proposition 3 *The probability measures* $(\mu_n)_{n\in\mathbb{N}^*}$ *on* $\mathcal{M}_{\mathcal{D}}(\mathcal{E})$ *converge for a sub set to a shift invariant probability measure* μ *on* $\mathcal{M}(\mathcal{E})$. *Moreover* μ*-almost every* $\Gamma \in \mathcal{M}(\mathcal{E})$ *satisfies the condition* (i′), (ii), (iii), (iv), *where* (i′) *is* (i′) $\Gamma^{(1)}$ *is in* $\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)$ *or* $\Gamma = 0$.

Moreover,

$$\mu(\Gamma^{(1)} \in \mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)) > 0. \tag{14}$$

Proof This proposition is the main result finding of this first part of the proof of Theorem 1. It is based on the four following lemmas. Lemma 1 allows to dominate uniformly the intensity of (μ_n) and obtain the convergence of (μ_n) to a measure μ. Next, we have to prove that μ satisfies the properties as claimed in the proposition. Lemma 2 controls the size of the cells in the tessellations during the convergence. Coupled with Lemma 3, which guarantees the local regularity of μ, we prove that μ satisfies the properties (i′), (ii), (iii), (iv). The property (14) is obtained thanks to the last Lemma 4.

Lemma 1 *There exists a constant* C_1 *such that for every* n *in* $\mathbb{N}^*$ *and every* $\Lambda \in \mathcal{B}(\mathbb{R}^2)$ *we have*

$$E_{\mu_n}(\Gamma^{(1)}(\Lambda)) \le C_1\lambda(\Lambda).$$

Proof μ_n is obviously shift invariant. So, it is sufficient to prove that μ_n has a σ-finite moment uniformly bounded for $n \in \mathbb{N}^*$. To do this, we can use the reduced Campbell measure of μ_n to identify the first moment. Since μ_n is the n-periodic Gibbs tessellation and the energy can be written as in (11) we have

$$C^{(n)!}_{\mu_{n^{(1)}}}(dx, d\gamma_{\Lambda_n}) = e^{-E^n(x,\gamma_{\Lambda_n})}\lambda_{\Lambda_n}(dx)\mu_{n^{(1)}}(d\gamma_{\Lambda_n}) \tag{15}$$

where the Campbell measure $C^{(n)!}_{\mu_{n^{(1)}}}$ is defined on $\Lambda_n \times \mathcal{M}(\Lambda_n)$ by

$$C^{(n)!}_{\mu_{n^{(1)}}}(f) = \int_{\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)}\int_{\Lambda_n} f(x, \gamma_{\Lambda_n} - \delta_x)\gamma(dx)\mu_{n^{(1)}}(d\gamma),$$

and $E^n(x, \gamma_{\Lambda_n})$ is the periodic energy of x in γ_{Λ_n}

$$E^n(x, \gamma_{\Lambda_n}) = E^n_{\Lambda_n}(\overline{p_n(\gamma_{\Lambda_n} + \delta_x)}) - E^n_{\Lambda_n}(\overline{p_n(\gamma_{\Lambda_n})}). \tag{16}$$

In fact, we have

$$E^n_{\Lambda_n}(\overline{p_n(\gamma)}) = \sum_{i=1}^{k} E^n(x_i, \delta_{x_1} + \cdots + \delta_{x_{i-1}}) \quad \text{for every } \gamma \in \mathcal{M}(\Lambda_n) \tag{17}$$

where $\gamma = \delta_{x_1} + \cdots + \delta_{x_n}$.

For the classical techniques concerning the calculus about the Campbell measure, we can see for example [13]. In particular, we can find the equation (15) for the Campbell measure of a Gibbs state. As above, it is easy to show that $|E^n(x, \gamma_{\Lambda_n})| \leq 2CM$. So, thanks to (15), the first moments of measures $(\mu_n)_{n\in\mathbb{N}^*}$ are uniformly bounded by $e^{2CM}\lambda$. The lemma is proved for the constant $C_1 = e^{2CM}$. □

In [8], Corollary A2.6.V p. 607 gives a characterization for the relative compacity of a set of radon measures on a locally compact space. A set (Γ_n) in $\mathcal{M}(\bar{\mathcal{E}})$ is relatively compact if

$$\forall \Lambda \in \mathcal{B}(\mathcal{E}),\ \exists C > 0,\ \forall n \in \mathbb{N}^*, \quad \Gamma_n(\Lambda) \leq C.$$

In our case, thanks to Lemma 1 we have

$$\forall \Lambda \in \mathcal{B}(\mathcal{E}),\ \exists C > 0,\ \forall n \in \mathbb{N}^*, \quad E_{\mu_n}(\Gamma(\Lambda)) \leq C.$$

Consequently, the family of probability measures (μ_n) is uniformly tight in $\mathcal{M}(\bar{\mathcal{E}})$. We can extract a sub set $(\mu_{\varphi(n)})$ converging weakly in $\mathcal{M}(\bar{\mathcal{E}})$ to a probability measure μ, this probability measure is clearly shift invariant.

Now, we have to prove some properties of this measure μ. Using the Skorokhod representation (see for example [4]) , we introduce some processes $(\tilde{\Gamma}_n)_{n\in\mathbb{N}^*}$ which converge almost surely to a process $\tilde{\Gamma}$ and such that μ (respectively μ_n) is the law of $\tilde{\Gamma}$ (respectively $\tilde{\Gamma}_n$). Therefore, it is easier to study μ through $\tilde{\Gamma}$ and $(\tilde{\Gamma}_n)_{n\in\mathbb{N}^*}$.

More precisely, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and processes $(\tilde{\Gamma}_n)_{n\in\mathbb{N}^*}$, $\tilde{\Gamma}$ from $\tilde{\Omega}$ to $\mathcal{M}(\bar{\mathcal{E}})$ such that for every $n \in \mathbb{N}^*$ the law of $\tilde{\Gamma}_n$ (respectively $\tilde{\Gamma}$) is μ_n (respectively μ). Moreover $(\tilde{\Gamma}_n)$ converges $\tilde{P}$-almost surely to $\tilde{\Gamma}$ for the metric $d_{\mathcal{M}(\bar{\mathcal{E}})}$ (we will skip until the end of the proof the notation $\mu_{\varphi(n)}$; we will just write μ_n).

For the moment, the only Delaunay property of $\tilde{\Gamma}_n$, which is kept for the limit $\tilde{\Gamma}$, is (ii). Let us begin with the analysis of $\tilde{\Gamma}$ with the following lemma.

Lemma 2 *$\tilde{P}$ almost surely $(\tilde{\Gamma}_n)$ satisfies*

$$\forall R > 0,\ \exists A > 0,\ \forall n \in \mathbb{N}^*,\ \forall X = \{x, y, z\} \in \tilde{\Gamma}_n^{(3)} \quad |x| \leq R \Rightarrow \max(|y|, |z|) \leq A. \tag{18}$$

Proof Let us suppose that (18) is false. This means that, with a strictly positive probability,

$$\exists R > 0,\ \forall A > 0,\ \exists n \in \mathbb{N}^*,\ \exists X = \{x, y, z\} \in \tilde{\Gamma}_n^{(3)} \quad |x| \leq R \quad \text{and} \quad \max(|y|, |z|) \geq A.$$

In other words, there exists $R > 0$ and a subset of triangles $(X_{\varphi(n)})_{n\in\mathbb{N}^*}$ such that for every $n \in \mathbb{N}^*$ $X_{\varphi(n)} = \{x_{\varphi(n)}, y_{\varphi(n)}, z_{\varphi(n)}\} \in \tilde{\Gamma}_{\varphi(n)}$, $x_{\varphi(n)} \leq R$ and the radius $r_{\varphi(n)}$ of the circumscribed ball of $X_{\varphi(n)}$ goes to infinity. This proves the existence of a half plane in $\mathbb{R}^2$ without point for the limit process $\tilde{\Gamma}$. But $\tilde{\Gamma}(B(0, R)) \geq 1$ since $\tilde{\Gamma}_{\varphi_n} B(0, R) \geq 1$ for every $n \in \mathbb{N}^*$. There is a contradiction here because a stationary point process, having a half plane without points, is necessary the process null. The lemma is proved. □

Now, thanks to this lemma, we know that $\tilde{\Gamma}$ satisfies the assumption (iii) $\tilde{P}$-almost surely. So, $\tilde{P}$-almost surely, every edge in $\tilde{\Gamma}$ is contained in two triangles. So the convex hull of $\tilde{\Gamma}$, $\langle\tilde{\Gamma}\rangle$ has no boundary and is equal to $\mathbb{R}^2$ or the empty set. It implies $\tilde{\Gamma}^{(1)}$ is equal to 0 or has no half plane without points. To prove (i), it must be shown that $\tilde{\Gamma}$ almost surely does not have four points on a same circle. It is a direct consequence of the following lemma which claims that the probability measure $\mu^{(1)}$ is locally absolutely continuous with respect to π.

Lemma 3 *$\mu^{(1)}$ is locally absolutely continuous with respect to π.*

Proof Equation (15) and the domination of $E^n(x, \gamma_{\Lambda_n})$ by the constant $2CM$ imply for every local bounded continuous function f and for n large enough,

$$C^{(n)!}_{\mu_{n^{(1)}}}(f) \leq e^{2CM} \lambda_{\Lambda_n} \otimes \mu_{n^{(1)}}(f).$$

Let n goes to infinity we have

$$C^{!}_{\mu^{(1)}}(f) \leq e^{2CM} \lambda \otimes \mu^{(1)}(f).$$

So $C_{\mu^{(1)}}$ is absolutely continuous with respect $\lambda \otimes \mu$. We deduce, thanks to a Theorem by Glotzl (Theorem 1 [10]), $\mu^{(1)}$ is a Gibbs point process for some potential. That implies the local absolute continuity of $\mu^{(1)}$ with respect to π. □

Thanks to this lemma we have many consequences for the process $\tilde{\Gamma}$. First of all, we validate that $\tilde{\Gamma}$ almost surely satisfies the property (i). Secondly, we show that every $X \in \tilde{\Gamma}$ is in $\mathcal{E}$ because, in principle, we know that X is in $\bar{\mathcal{E}}$. Moreover, the property (iv) is also satisfied by $\tilde{\Gamma}$. Concerning the last property in Proposition 3, it is a direct consequence of this following lemma.

Lemma 4 *There exists a constant $C_2 > 0$ such that for every n in $\mathbb{N}^*$ and every $\Lambda \in \mathcal{B}(\mathbb{R}^2)$ we have*

$$E_{\mu_n}(\Gamma^{(1)}(\Lambda)) \geq C_2 \lambda(\Lambda).$$

Proof It is exactly the same proof as in Lemma 1. □

The proof of Proposition 3 is finished. □

Let us remark that we are not sure that $\mu(\Gamma^{(1)} \in \mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)) = 1$, but the definition of Gibbs Delaunay tessellations implies this property. In fact, the good candidate for the Gibbs Delaunay tessellation is the probability measure $\boldsymbol{\mu} := \mu(.|\Gamma^{(1)} \in \mathcal{M}_{\mathcal{D}}(\mathbb{R}^2))$, we will see this later.

Second step, the DLR equations:

The DLR equations (9) are essentially based on the continuity of the kernels (Ξ_Λ). It is however easy to see that $\Xi_\Lambda(., f)$ is not continuous everywhere because the potential V is not continuous and the Delaunay structure is not stable near a configuration having almost four points on a same circle. To solve this problem, we prove only the continuity of the kernels for μ-almost every Γ. Lemma 3 is crucial.

We begin to analyse the continuity of the kernels (Ξ_Λ) in Lemma 5. Next, to prove the DLR equations, we introduce equilibrium equations (23) for the measures (μ_n) which are inspired by the compatibility equations (8). Since (μ_n) are the periodic Gibbs tessellations on (Λ_n), we have to define the periodic kernels (Ξ^n_Λ). Afterwards, we have to control these equations when n goes to infinity which is equivalent to Λ' goes to $\mathbb{R}^2$ in (8). We directly obtain the DLR equations.

Let us give a precise definition of the set for which we have the continuity of the kernels. Let $(\Delta_m)_{m\in\mathbb{N}^*}$ a set of bounded sets in $\mathbb{R}^2$ growing to $\mathbb{R}^2$ and having smooth boundaries

$(\partial\Delta_m)_{n\in\mathbb{N}^*}$; for example we can take $\Delta_m = B(0,m)$. We define

$$\mathbf{M} = \left\{ \begin{array}{l} \Gamma \in \mathcal{M}(\mathcal{E}) \text{ such that} \\ \quad \text{(a)} \;\; \Gamma = 0 \text{ or } \Gamma \in \mathcal{M}^\infty_\mathcal{D}(\mathcal{E}) \\ \quad \text{(b)} \;\; \forall m \in \mathbb{N}^*, \;\; \Gamma^{(1)}(\partial\Delta_m) = 0 \\ \quad \text{(c)} \;\; \forall m \in \mathbb{N}^*, \;\; \text{any } X \in \Gamma^{(2)} \text{ is not tangential to } \partial\Delta_m \\ \quad \text{(d)} \;\; V \text{ is continuous for every subconfiguration } \Gamma' \text{ in } \Gamma \end{array} \right\}. \tag{19}$$

We note $\mathbf{M}^* = \mathbf{M} - \{0\}$. Thanks to Proposition 3, Lemma 3 and the assumption (**R**) we claim

$$\mu(\mathbf{M}) = 1.$$

We want to prove the continuity of the kernels $\Xi_{\Delta_m}(\Gamma,.)$ at each Γ in $\mathbf{M}$, so we have to extend the definition of $\Xi_\Lambda(\Gamma,.)$ for $\Gamma = 0$. We put $\Xi_\Lambda(0,.) = \delta_0$.

Lemma 5 *For every $m \in \mathbb{N}^*$, for every bounded continuous function f from $\mathcal{M}(\mathcal{E})$ to $\mathbb{R}$, the function from $\mathcal{M}_\mathcal{D}(\mathcal{E}) \cup \{O\}$ to $\mathbb{R}$: $\Gamma \longrightarrow \Xi_{\Delta_m}(\Gamma, f)$ is continuous at every $\Gamma \in \mathbf{M}$.*

Proof Let m be in $\mathbb{N}^*$ and f a bounded continuous function from $\mathcal{M}(\mathcal{E})$ to $\mathbb{R}$.

Beginning with the trivial case where $\Gamma = 0$. Let (Γ_n) be a set of elements in $\mathcal{M}_\mathcal{D}(\mathcal{E})$ such that Γ_n goes to 0 when n goes to infinity. Then, for n large enough, $\Gamma_n^{(1)}$ has no point in Δ_m. So, $S_{\Delta_m}(\Gamma_n)$ is empty. Therefore $\Xi_{\Delta_m}(\Gamma_n,.) = \delta_{\Gamma_n}$ and $\lim_{n\to\infty} \Xi_{\Delta_m}(\Gamma_n, f) = \Xi_{\Delta_m}(0, f) = f(0)$.

Now, let us suppose $\Gamma \in \mathcal{M}_\mathcal{D}(\mathcal{E}) \cap \mathbf{M}^*$. It is easy to see that the function from $\mathcal{M}_\mathcal{D}(\mathcal{E})$ to $\mathcal{M}(\mathcal{E})$: $\Gamma \longrightarrow \Gamma_{\Delta_m^c}$ is continuous at each $\Gamma \in \mathbf{M}^*$ because there is no boundary problem thanks to the points b) and c). Let us not forget that the topology on $\mathcal{M}(\mathcal{E})$ is induced by the metric $d_{\mathcal{M}(\mathcal{E})}$. We deduce that the function from $\mathcal{M}_\mathcal{D}(\mathcal{E}) \times \mathcal{M}(\mathbb{R}^2)$ to $\mathcal{M}_\mathcal{D}(\mathcal{E})$: $(\Gamma,\gamma) \longrightarrow \Upsilon(\Gamma,\gamma,\Delta_m)$ is continuous for every $\Gamma \in \mathbf{M}^*$ and for π-almost every $\gamma \in \mathcal{M}(\mathbb{R}^2)$. Now, thanks to the assumption d) in the definition of $\mathbf{M}$, we claim that the function from $\mathcal{M}_\mathcal{D}(\mathcal{E}) \times \mathcal{M}(\mathbb{R}^2)$ to $\mathcal{M}_\mathcal{D}(\mathcal{E})$: $(\Gamma,\gamma) \longrightarrow E_{\Delta_m}(\Upsilon(\Gamma,\gamma,\Delta_m))$ is continuous at every $\Gamma \in \mathbf{M}^*$ and π-almost every $\gamma \in \mathcal{M}(\mathbb{R}^2)$.

We can now see the continuity of the kernels $(\Xi_{\Delta_m})_{n\in\mathbb{N}^*}$ precisely. Let us remember the definition: for every $\Gamma \in \mathcal{M}^\infty_\mathcal{D}(\mathcal{E})$ and every test function f from $\mathcal{M}(\mathcal{E})$ to $\mathbb{R}$ we have

$$\Xi_\Lambda(\Gamma, f) = \int_{\mathcal{M}(\mathbb{R}^2)} f(\Upsilon(\Gamma,\gamma,\Lambda)) \frac{1}{Z_\Lambda(\Gamma)} e^{-E_\Lambda(\Upsilon(\Gamma,\gamma,\Lambda))} \pi_{S_\Lambda(\Gamma)}(d\gamma),$$

where $Z_\Lambda(\Gamma)$ is

$$Z_\Lambda(\Gamma) = \int_{\mathcal{M}(\mathbb{R}^2)} e^{-E_\Lambda(\Upsilon(\Gamma,\gamma,\Lambda))} \pi_{S_\Lambda(\Gamma)}(d\gamma)$$

if $S_\Lambda(\Gamma) \neq \emptyset$ and otherwise

$$\Xi_\Lambda(\Gamma, f) = f(\Gamma).$$

Let (Γ_n) be a set of elements in $\mathcal{M}_\mathcal{D}(\mathcal{E})$ such that (Γ_n) goes to Γ when n goes to infinity. Let us show that $\Xi_{\Delta_m}(\Gamma_n, f)$ goes to $\Xi_{\Delta_m}(\Gamma, V)$. If $S_{\Delta_m}(\Gamma) = \emptyset$ the proof is the same as above in the case $\Gamma = 0$. Otherwise, thanks to the dominated convergence Theorem by Lebesgue, we have

$$\lim_{n\to\infty} Z_{\Delta_m}(\Gamma_n) = \lim_{n\to\infty}\int_{\mathcal{M}(\mathbb{R}^2)} e^{-E_{\Delta_m}(\Upsilon(\Gamma_n,\gamma,\Delta_m))}\pi_{\mathcal{S}_{\Delta_m}(\Gamma_n)}(d\gamma)$$
$$= \int_{\mathcal{M}(\mathbb{R}^2)} e^{-E_{\Delta_m}(\Upsilon(\Gamma,\gamma,\Delta_m))}\pi_{\mathcal{S}_{\Delta_m}(\Gamma)}(d\gamma)$$
$$= Z_{\Delta_m}(\Gamma).$$

The uniform domination comes from the uniform stability constant for the potential V(see the beginning of the proof in Proposition 3). The simple convergence of the function inside the integral comes from the continuity of the function $E_{\Delta_m}(\Upsilon(.,.,\Delta_m))$. With the same argument, it is easy to see that

$$\lim_{n\to\infty} \Xi_{\Delta_m}(\Gamma_n, f) = \Xi_{\Delta_m}(\Gamma, f).$$

The lemma is consequently proved. □

Since the measures (μ_n) are the periodic Gibbs Delaunay tessellations on Λ_n, let us give an equilibrium equation for μ_n. We have to introduce some notations. We note for every $\Gamma \in \mathcal{M}_{\mathcal{D}}(\mathcal{E})$, every $\Lambda \in \mathcal{B}(\mathbb{R}^2)$

$$\Gamma^n_{\Lambda^c} = \sum_{\substack{X\in\Gamma \\ \exists Y\in\Gamma^{(3)},\ X\subset Y \\ \langle Y\rangle\cap p_n(\Lambda)^c\neq\emptyset}} \delta_X \tag{20}$$

where $p_n(\Lambda)$ is the set $\bigcup_{x\in\Lambda}\{x+h, h\in 2n\mathbb{Z}^d\}$. Let us define the set $\mathcal{S}^n_\Lambda(\Gamma)$ in $\mathbb{R}^2$

$$\mathcal{S}^n_\Lambda(\Gamma) = \mathbb{R}^2\backslash\left(\bigcup_{X\in(\Gamma^n_{\Lambda^c})^{(3)}} \bar{B}(X)\right) \tag{21}$$

and the function Υ^n

$$\Upsilon^n\colon \mathcal{M}_{\mathcal{D}}(\mathcal{E})\times\mathcal{M}(\mathbb{R}^2)\times\mathcal{B}(\mathbb{R}^2) \longrightarrow \mathcal{M}_{\mathcal{D}}(\mathcal{E}),$$
$$(\Gamma,\gamma,\Lambda) \longrightarrow \overline{(\Gamma^{(1)}_{\mathcal{S}^n_\Lambda(\Gamma)^c} + \gamma_{\mathcal{S}^n_\Lambda(\Gamma)})}.$$

If $\Gamma^{(1)}_{\mathcal{S}^n_\Lambda(\Gamma)^c} + \gamma_{\mathcal{S}^n_\Lambda(\Gamma)}$ is not in $\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)$, we put $\Upsilon^n(\Gamma,\gamma,\Lambda) = \Gamma$.

Now, we define the following kernels Ξ^n_Λ on $\mathcal{M}^\infty_{\mathcal{D}}(\mathcal{E})\times\mathcal{P}(\mathcal{M}^\infty_{\mathcal{D}}(\mathcal{E}))$. For every n-periodic Γ in $\mathcal{M}^\infty_{\mathcal{D}}(\mathcal{E})$ and every test function f from $\mathcal{M}^\infty_{\mathcal{D}}(\mathcal{E})$ to $\mathbb{R}$ we define

$$\Xi^n_\Lambda(\Gamma, f) = \int_{\mathcal{M}(\mathbb{R}^2)} f(\Upsilon^n(\Gamma, p_n(\gamma),\Lambda))\frac{1}{Z^n_\Lambda(\Gamma)} \times e^{-E^n_\Lambda(\Upsilon^n(\Gamma,p_n(\gamma),\Lambda))}\pi_{\mathcal{S}_\Lambda(\Gamma)\cap\Lambda_n}(d\gamma), \tag{22}$$

where $Z^n_\Lambda(\Gamma)$ is the normalization constant

$$Z^n_\Lambda(\Gamma) = \int_{\mathcal{M}(\mathbb{R}^2)} e^{-E^n_\Lambda(\Upsilon^n(\Gamma,p_n(\gamma),\Lambda))}\pi_{\mathcal{S}_\Lambda(\Gamma)\cap\Lambda_n}(d\gamma).$$

The kernels $(\Xi_\Lambda^n)_{\Lambda\in\mathcal{B}(\mathbb{R}^2)}$ are just the periodic version of the kernels $(\Xi_\Lambda)_{\Lambda\in\mathcal{B}(\mathbb{R}^2)}$. Thanks to these kernels, we can write the following equilibrium equation for μ_n which is just the compatibility of the kernels $(\Xi_\Lambda^n)_{\Lambda\in\mathcal{B}(\mathbb{R}^2)}$ on Λ_n (see Proposition 2 for a similar proof). For every bounded set Λ in $\mathcal{B}(\mathbb{R}^2)$ and every bounded continuous function f from $\mathcal{M}_\mathcal{D}(\mathcal{E})$ to $\mathbb{R}$ we have

$$\int_{\mathcal{M}_\mathcal{D}(\mathcal{E})} f(\Gamma)\mu_n(d\Gamma) = \int_{\mathcal{M}_\mathcal{D}(\mathcal{E})}\int_{\mathcal{M}_\mathcal{D}(\mathcal{E})} f(\Gamma')\Xi_\Lambda^n(\Gamma, d\Gamma')\mu_n(d\Gamma). \tag{23}$$

Let us now prove the DLR equations (9) for μ with $\Lambda = \Delta_m$, $m \in \mathbb{N}^*$. In the following limits, we use both indirectly and clearly the locality of the Delaunay tessellation.

$$\begin{aligned}
&\int_{\mathcal{M}_\mathcal{D}(\mathcal{E})} f(\Gamma)\mu(d\Gamma)\\
&\quad= \lim_{n\to\infty}\int_{\mathcal{M}_\mathcal{D}(\mathcal{E})} f(\Gamma)\mu_n(d\Gamma)\\
&\quad= \lim_{n\to\infty}\int_{\mathcal{M}_\mathcal{D}(\mathcal{E})}\int_{\mathcal{M}_\mathcal{D}(\mathcal{E})} f(\Gamma')\Xi_{\Delta_m}^n(\Gamma, d\Gamma')\mu_n(d\Gamma)\\
&\quad= \lim_{n\to\infty}\int_{\mathcal{M}_\mathcal{D}(\mathcal{E})} \Xi_{\Delta_m}(\Gamma, f)\mu_n(d\Gamma)\\
&\qquad+ \lim_{n\to\infty}\int_{\mathcal{M}_\mathcal{D}(\mathcal{E})} (\Xi_{\Lambda_m}^n(\Gamma, f) - \Xi_{\Delta_m}(\Gamma, f))\mu_n(d\Gamma)\\
&\quad= \int_{\mathcal{M}_\mathcal{D}(\mathcal{E})} \Xi_{\Delta_m}(\Gamma, f)\mu(d\Gamma) + \lim_{n\to\infty}\int (\Xi_{\Delta_m}^n(\tilde{\Gamma}_n, f) - \Xi_{\Delta_m}(\tilde{\Gamma}_n, f))d\tilde{P}.
\end{aligned} \tag{24}$$

In the last equality, the first part comes from Lemma 5 and the fact $\mu(\mathbf{M}) = 1$. The second part comes from the representation of the weak convergence introduced just before Lemma 2.

To prove that the second part of the last equality is null, let us show that the function $\Xi_{\Delta_m}^n(\tilde{\Gamma}_n, f) - \Xi_{\Delta_m}(\tilde{\Gamma}_n, f)$ goes almost surely to 0, when n goes to infinity. The dominated convergence Theorem completes the proof since this function is dominated by $2\|f\|_\infty$.

If $(\tilde{\Gamma}_n)$ goes to 0, then both kernels tested on f go to $f(0)$ and the difference goes to 0. Now, if $(\tilde{\Gamma}_n)$ goes to $\tilde{\Gamma} \neq 0$. Since f is continuous for the metric $d_{\mathcal{M}(\mathcal{E})}$, f is the uniform limit of a set of local continuous functions. So, it is sufficient to prove the convergence for a local function f. Since $\tilde{\Gamma}_n$ goes to $\tilde{\Gamma} \neq 0$ when n goes to infinity, then for n large enough, the kernels $\Xi_{\Delta_m}^n(\tilde{\Gamma}_n, f)$ and $\Xi_{\Delta_m}(\tilde{\Gamma}_n, f)$ are equal; the difference is null. So, the equality (24) becomes

$$\int_{\mathcal{M}_\mathcal{D}(\mathcal{E})} f(\Gamma)\mu(d\Gamma) = \int_{\mathcal{M}_\mathcal{D}(\mathcal{E})} \Xi_{\Delta_m}(\Gamma, f)\mu(d\Gamma). \tag{25}$$

It is exactly the equation DLR (9) for $\Lambda = \Delta_m$. To obtain the equilibrium equations (9) for any Λ, it is just a consequence of the DLR equation (25) for some Δ_m containing Λ and the compatibility of kernels (8). Now, let us point out that μ is not a Gibbs Delaunay tessellation because $\mu(\mathcal{M}_\mathcal{D}(\mathcal{E})) = \mu(\Gamma \neq 0)$ may be not equal to 1. Thanks to Proposition 3, we know that $\mu(\Gamma \neq 0) > 0$. The Gibbs Delaunay tessellation is in fact the measure $\boldsymbol{\mu} = \mu(.|\Gamma \neq 0)$.

Let us verify that $\boldsymbol{\mu}$ satisfies the DLR equations (9). From (25)

$$\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} f(\Gamma)(\mu(d\Gamma|\Gamma \neq 0)\mu(\Gamma \neq 0) + \mu(d\Gamma|\Gamma = 0)\mu(\Gamma = 0))$$
$$= \int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} \Xi_\Lambda(\Gamma, f)(\mu(d\Gamma|\Gamma \neq 0)\mu(\Gamma \neq 0) + \mu(d\Gamma|\Gamma = 0)\mu(\Gamma = 0)).$$

If we expand, that becomes

$$\mu(\Gamma \neq 0) \int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} f(\Gamma)\boldsymbol{\mu}(d\Gamma) + f(0)\mu(\Gamma = 0)$$
$$= \mu(\Gamma \neq 0) \int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} \Xi_\Lambda(\Gamma, f)\boldsymbol{\mu}(d\Gamma) + f(0)\mu(\Gamma = 0)$$

and so

$$\int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} f(\Gamma)\boldsymbol{\mu}(d\Gamma) = \int_{\mathcal{M}_{\mathcal{D}}(\mathcal{E})} \Xi_\Lambda(\Gamma, f)\boldsymbol{\mu}(d\Gamma).$$

Theorem 1 is proved. □

4 Existence of Gibbs Tessellations with Geometric Hardcore Conditions

In this part, we propose introducing a geometric hardcore condition on the potential V for small and large cells. Indeed, we put the energy of a cell equal to plus infinity if this cell is small or large enough. Let us remark that the hardcore condition about the small triangle is classical because it is almost equivalent to the hardcore condition between the points (hard-ball condition) but the second condition about the large cells is absolutely not a classical point hardcore condition. It depends on the geometry of the tessellation and is not hereditary. It means that the following situation is possible: $E_\Lambda(\gamma) = +\infty$ and $E_\Lambda(\gamma + \delta_x) < \infty$. Moreover, the interaction is not quasilocal because it goes to infinity when the size of the cell goes to infinity.

The Gibbs Delaunay tessellations, which we construct in this section, do not have (with probability one) small and large cells. It is an interesting result for the stochastic geometry point of view. Indeed, in [15], the author proposes a connection between the Gibbs point process and the Delaunay Tessellations to control the size of cells. This is exactly what we do by constructing more regular infinite random Delaunay tessellations.

Let us now introduce precisely the hardcore interaction for the potential V.

(HC): The hardcore condition
There exists strictly positive constants r and R such that for all X in $\mathcal{S}_p(\mathcal{E})$

$$V(X) = +\infty \iff \begin{array}{l} X \text{ is an edge with a length smaller or equal to } r \\ \text{or } X \text{ is a triangle with the radius of } B(X) \text{ greater or equal to } R. \end{array}$$

In the following theorem we suppose that V satisfies (HC) and (R) and there is no contradiction. Indeed, (HC) implies that V is discontinuous for every X in $\mathcal{E}^{(2)}$ having a length equal to r, or every X in $\mathcal{E}^{(3)}$ with the radius of $B(X)$ equal to R. In fact, it is not incompatible with (R) because these configurations $\bar{\pi}$-almost surely do not appear.

Now let us give the theorem with a geometric hardcore interaction.

Theorem 2 *There exists a Gibbs Delaunay tessellation for a potential V which satisfies* (TI), (HC), (B) *and* (R) *with $R > r$.*

Proof Let us begin by justifying that the kernels $(\Xi_\Lambda)_{\Lambda\in\mathcal{B}(\mathbb{R}^2)}$ are well defined. The problem is to prove that $0 < Z_\Lambda(\Gamma) < +\infty$ for every $\Gamma \in \mathcal{M}^\infty_\mathcal{D}(\mathcal{E})$ such that $S_\Lambda(\Gamma) \neq \emptyset$. By construction, $S_\Lambda(\Gamma)$ is an opened set in $\mathbb{R}^2$. Moreover the condition (HC) implies that the set $\mathcal{M}^\infty_\mathcal{D}(\mathcal{E})$ is locally an open set in $\mathcal{M}(\mathcal{E})$. It means that for every configuration $\Gamma \in \mathcal{M}^\infty_\mathcal{D}(\mathcal{E})$ we can locally perturb the configuration Γ such that this perturbation is still in $\mathcal{M}^\infty_\mathcal{D}(\mathcal{E})$. So, it is clear that $Z_\Lambda(\Gamma) > 0$.

Now let us prove the second inequality by using a classical stability argument. Let Γ' be in $\in \mathcal{M}_\mathcal{D}(\mathcal{E})$. Either $E_\Lambda(\Gamma') = +\infty$ and so $e^{-E_\Lambda(\Gamma')} = 0$, or $\Gamma' \in \mathcal{M}^\infty_\mathcal{D}(\mathcal{E})$ and, thanks to the assumption (HC), we prove that every triangle X in $\Gamma'^{(3)}$ has a radius of $B(X)$ smaller than R and has the lengths of its sides greater than r. This implies the existence of a uniform minimal angle $\alpha_0 > 0$ for every triangle in $\Gamma'^{(3)}$. With the same argument, in the proof of the boundness of $Z_\Lambda(\Gamma)$ in Theorem 1, we conclude there exists a constant C_Λ such that $|E_\Lambda(\Gamma')| < C_\Lambda \Gamma'^{(1)}(\Lambda)$. The potential is stable and so $Z_\Lambda(\Gamma) < +\infty$.

Now, the scheme of the proof of Theorem 2, is exactly the same as in Theorem 1. We just have to prove that the Lemmas 1, 3 and 4 are still true in the case where V satisfies (TI), (HC), (B) and (R). It can be seen that the Lemmas 1, 4 are obviously true if (HC) is satisfied.

The difficulty is essentially concentrated in the following Lemma 6 which is the analogue of Lemma 3. Since the interaction V is not hereditary, we can not define the energy of point x in a configuration γ. Therefore, the classical tools for Gibbs point processes, can not be used here. The proof in Lemma 3 is based on the Campbell measure via Glotzl's Theorems. So, in this section, we have to make a direct and more complicate proof of the local absolute continuity of μ.

Lemma 6 *Under the assumptions* (TI), (R), (B) *and* (HC) *with* ($R > r$), $\mu^{(1)}$ *is locally absolutely continuous with respect to π.*

Proof The proof of this lemma is long and technical. So let us begin by giving an outline. Let Λ be a bounded set in $\mathbb{R}^2$. We want to show that the projection of μ on Λ is absolutely continuous with respect to λ. Thanks to Lemma 7 and some calculus on the kernels, we have for every regular function f and every $\varepsilon > 0$

$$\int f(\Gamma^{(1)}_\Lambda)\mu(d\Gamma) \leq K\left(\frac{1}{\varepsilon}\int f(\gamma)\pi_\lambda(d\gamma) + \varepsilon\|f\|_\infty\right), \tag{26}$$

where K is a fixed constant. So, we can deduce easily that for every Borel set A in $\mathcal{M}(\Lambda)$ such that $\pi_\Lambda(A) = 0$, we have $\mu^{(1)}(A) = 0$. Using the Radon Nikodym Theorem, we can complete the proof. Now let us give the precise calculus.

Let Λ and Λ' be bounded sets in $\mathbb{R}^2$ such that $\Lambda \subset \Lambda'$ and such that

$$\forall x \in \Lambda,\ \forall y \in \Lambda'^c \quad |x - y| > 2R. \tag{27}$$

Let f_Λ be a bounded continuous function from $\mathcal{M}_\mathcal{D}(\mathbb{R}^2)$ to $\mathbb{R}$ measurable for the sigma field of $\mathcal{M}(\Lambda)$. Since (μ_n) converges weakly to μ (see Proposition 3), we have

$$\int_{\mathcal{M}(\bar{\mathcal{E}})} f_\Lambda(\Gamma^{(1)})\mu(d\Gamma) = \lim_{n\to\infty}\int_{\mathcal{M}^\infty_\mathcal{D}(\mathcal{E})} f_\Lambda(\Gamma^{(1)})\mu_n(d\Gamma). \tag{28}$$

For each $n \in \mathbb{N}^*$, μ_n is a solution of the periodic equilibrium equation (23). So

$$\int_{\mathcal{M}_{\mathcal{D}}^{\infty}(\mathcal{E})} f_\Lambda(\Gamma^{(1)})\mu_n(d\Gamma) = \int_{\mathcal{M}_{\mathcal{D}}^{\infty}(\mathcal{E})} \int_{\mathcal{M}_{\mathcal{D}}^{\infty}(\mathcal{E})} f_\Lambda(\Gamma'^{(1)})\Xi^n_{\Lambda'}(\Gamma, d\Gamma')\mu_n(d\Gamma). \tag{29}$$

Now we want to substitute $\Xi^n_{\Lambda'}$ with $\Xi_{\Lambda'}$. Using the same techniques as with equation (24), we find

$$\begin{aligned}\int_{\mathcal{M}(\mathcal{E})} f_\Lambda(\Gamma^{(1)})\mu(d\Gamma) &= \lim_{n\to\infty} \int_{\mathcal{M}_{\mathcal{D}}^{\infty}(\mathcal{E})} \int_{\mathcal{M}_{\mathcal{D}}^{\infty}(\mathcal{E})} f_\Lambda(\Gamma'^{(1)})\Xi_{\Lambda'}(\Gamma, d\Gamma')\mu_n(d\Gamma)\\ &= \lim_{n\to\infty} \int_{\tilde{\Omega}} \int_{\mathcal{M}_{\mathcal{D}}^{\infty}(\mathcal{E})} f_\Lambda(\Gamma'^{(1)})\Xi_{\Lambda'}(\tilde{\Gamma}_n, d\Gamma')d\tilde{P}. \end{aligned}\tag{30}$$

To simplify the calculus, we fix, for the moment, $\tilde{P}$-a.s. $(\tilde{\Gamma}_n)$ converging to $\tilde{\Gamma}$. We have

$$\begin{aligned}&\int_{\mathcal{M}_{\mathcal{D}}^{\infty}(\mathcal{E})} f_\Lambda(\Gamma'^{(1)})\Xi_{\Lambda'}(\tilde{\Gamma}_n, d\Gamma')d\tilde{P}\\ &\quad= \int_{\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)} f_\Lambda(\gamma)\frac{e^{-E_{\Lambda'}(\Upsilon(\tilde{\Gamma}_n,\gamma,\Lambda'))}}{Z_{\Lambda'}(\tilde{\Gamma}_n)}\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}(d\gamma)\end{aligned}\tag{31}$$

where

$$Z_{\Lambda'}(\tilde{\Gamma}_n) = \int_{\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)} e^{-E_{\Lambda'}(\Upsilon(\tilde{\Gamma}_n,\gamma,\Lambda'))}\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}(d\gamma).$$

Let us now introduce both following sets included in $\mathcal{M}(S_{\Lambda'}(\tilde{\Gamma}_n))$

$$\mathcal{K}(\tilde{\Gamma}_n) = \{\gamma \in \mathcal{M}(S_{\Lambda'}(\tilde{\Gamma}_n)) \text{ such that } E_{\Lambda'}(\Upsilon(\tilde{\Gamma}_n, \gamma, \Lambda')) < +\infty\} \tag{32}$$

and for $\varepsilon > 0$

$$\mathcal{K}^\varepsilon(\tilde{\Gamma}_n) = \left\{\gamma \in \mathcal{K}(\tilde{\Gamma}_n) \text{ such that } \int_{\mathcal{M}(\Lambda)} \mathbb{1}_{\mathcal{K}(\tilde{\Gamma}_n)}(\gamma_{\Lambda'-\Lambda} + \gamma')\pi_\Lambda(d\gamma') \geq \varepsilon\right\}. \tag{33}$$

$\mathcal{K}(\tilde{\Gamma}_n)$ is the support of $\Gamma'^{(1)}$ in $S_{\Lambda'}(\tilde{\Gamma}_n)$ under the probability measure $\Xi_{\Lambda'}(\tilde{\Gamma}_n, .)$. $\mathcal{K}^\varepsilon(\tilde{\Gamma}_n)$ represents the good configurations γ in $\mathcal{K}(\tilde{\Gamma}_n)$. Indeed, these configurations have a subset in $\mathcal{M}(\Lambda)$, with probability bigger than ε under π_Λ, to spread out. We have the following upperboundedness which controls the difference between $\mathcal{K}(\tilde{\Gamma}_n)$ and $\mathcal{K}^\varepsilon(\tilde{\Gamma}_n)$.

$$\begin{aligned}&\Bigg|\int_{\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)} f_\Lambda(\gamma)\frac{e^{-E_{\Lambda'}(\Upsilon(\tilde{\Gamma}_n,\gamma,\Lambda'))}}{Z_{\Lambda'}(\tilde{\Gamma}_n)}\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}(d\gamma)\\ &\qquad - \int_{\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)} f_\Lambda(\gamma)\mathbb{1}_{\mathcal{K}^\varepsilon(\tilde{\Gamma}_n)}(\gamma)\frac{e^{-E_{\Lambda'}(\Upsilon(\tilde{\Gamma}_n,\gamma,\Lambda'))}}{Z_{\Lambda'}(\tilde{\Gamma}_n)}\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}(d\gamma)\Bigg|\\ &\quad\leq \int_{\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)} \left|f_\Lambda(\gamma)\frac{e^{-E_{\Lambda'}(\Upsilon(\tilde{\Gamma}_n,\gamma,\Lambda'))}}{Z_{\Lambda'}(\tilde{\Gamma}_n)}\right|(1-\mathbb{1}_{\mathcal{K}^\varepsilon(\tilde{\Gamma}_n)}(\gamma))\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}(d\gamma)\end{aligned}$$

$$\leq \|f_\Lambda\|_\infty \frac{\sup_{\gamma\in\mathcal{K}(\tilde{\Gamma}_n)} e^{-E_{\Lambda'}(\Upsilon(\tilde{\Gamma}_n,\gamma,\Lambda'))}}{\inf_{\gamma\in\mathcal{K}(\tilde{\Gamma}_n)} e^{-E_{\Lambda'}(\Upsilon(\tilde{\Gamma}_n,\gamma,\Lambda'))}} \frac{\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}(\mathcal{K}(\tilde{\Gamma}_n)-\mathcal{K}^\varepsilon(\tilde{\Gamma}_n))}{\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}((\mathcal{K}(\tilde{\Gamma}_n))}$$

$$\leq e^{2C_{\Lambda'}}\|f_\Lambda\|_\infty \frac{\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}(\mathcal{K}(\tilde{\Gamma}_n)-\mathcal{K}^\varepsilon(\tilde{\Gamma}_n))}{\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}((\mathcal{K}(\tilde{\Gamma}_n))} \tag{34}$$

where $C_{\Lambda'}$ is an uniform bound for the function $|E_{\Lambda'}(\Upsilon(\tilde{\Gamma}_n,.,\Lambda'))|$ on $\mathcal{K}(\tilde{\Gamma}_n)$, which exists thanks to the assumptions (HC) and (B). $C_{\Lambda'}$ does not depend on $(\tilde{\Gamma}_n)$. To control the last term in the inequality (34), we have the following lemma.

Lemma 7 *There exists a constant $\rho > 0$ such that for every $\varepsilon > 0$*

$$\frac{\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}(\mathcal{K}(\tilde{\Gamma}_n)-\mathcal{K}^\varepsilon(\tilde{\Gamma}_n))}{\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}((\mathcal{K}(\tilde{\Gamma}_n))} \leq \rho\varepsilon.$$

Proof Let γ be in $\mathcal{K}_{\Lambda'}(\tilde{\Gamma}_n) - \mathcal{K}^\varepsilon_{\Lambda'}(\tilde{\Gamma}_n)$. The problem of such a configuration is that the points of γ inside Λ are almost frozen in the sense where $\pi_\Lambda(\gamma'_\Lambda|\gamma_{\Lambda'-\Lambda}+\gamma'_\Lambda \in \mathcal{K}(\tilde{\Gamma}_n)) \leq \varepsilon$. We want to construct a new configuration which is equal to $\gamma_{\Lambda'-\Lambda}$ plus a second part $\hat{\gamma} = \delta\hat{x}_1 + \cdots + \delta\hat{x}_l$, such that this configuration $\gamma_{\Lambda'-\Lambda}+\hat{\gamma}$ is in $\mathcal{K}^\varepsilon(\tilde{\Gamma}_n)$. Let us define a function from $\mathcal{K}(\tilde{\Gamma}_n)$ to $\mathcal{M}(\mathcal{S}_{\Lambda'}(\tilde{\Gamma}_n))$: $\gamma \longmapsto \hat{\gamma} = \hat{y}_1 + \cdots + \hat{y}_l$ such that for all $(h_i)_{1\leq i\leq l}$ in $B(0,\frac{\delta}{3})^l$ we have

$$\gamma_{\Lambda'-\Lambda} + \delta_{\hat{y}_1+h_1} + \cdots + \delta_{\hat{y}_l+h_l} \quad \text{is in } \mathcal{K}(\tilde{\Gamma}_n), \tag{35}$$

where δ is equal to $\frac{R-r}{2}$. Let us remark that, if (35) is satisfied, $\gamma_{\Lambda'-\Lambda}+\hat{\gamma}$ is in $\mathcal{K}^\varepsilon(\tilde{\Gamma}_n)$ for ε small enough. How can this configuration $\hat{\gamma}$ be constructed?

In fact, we construct $\hat{\gamma}$ recursively. We fix $\hat{\gamma}=0$ and we test if $\hat{\gamma}$ satisfies the following assumption (H)

$$\forall x \in \mathcal{S}_{\Lambda'}(\tilde{\Gamma}_n), \quad \Upsilon(\tilde{\Gamma}_n, \gamma_{\Lambda'-\Lambda}+\hat{\gamma}, \Lambda')^{(1)}(B(x, r+\delta)) > 0.$$

If this is the case, we stop the recursive process. If this is not the case, we choose an arbitrary point $\hat{x}$ in $\mathcal{S}_{\Lambda'}(\tilde{\Gamma}_n)$, such that $\Upsilon(\tilde{\Gamma}_n, \gamma_{\Lambda'-\Lambda}+\hat{\gamma}, \Lambda')^{(1)}(B(x, r+\delta)) = 0$, and we put $\hat{\gamma} := \hat{\gamma} + \delta_{\hat{x}}$. Now, we test the assumption (H) again for this new $\hat{\gamma}$. If (H) is true, we stop here otherwise we choose another point $\hat{x}'$ as above and we put $\hat{\gamma} := \hat{\gamma} + \delta_{\hat{x}'}$. We go on like this until the process stops which is always the case since $\mathcal{S}_{\Lambda'}(\tilde{\Gamma}_n)$ is bounded and the minimal distance between two points in $\hat{\gamma}$ is larger than r. The application $\gamma \to \hat{\gamma}$ must be measurable, so we choose the point $\hat{x}$ with a measurable function. We can use, for example, the lexicographic order in $\mathbb{R}^2$.

Now let us prove that $(\gamma,\hat{\gamma})$ satisfies (35). Let us write $\hat{\gamma} = \delta\hat{y}_1 + \cdots + \delta\hat{y}_l$ and for all $(h_i)_{1\leq i\leq l}$ in $B(0,\frac{\alpha}{3})^l$, we note $\gamma' = \gamma_{\Lambda'-\Lambda} + \delta_{\hat{y}_1+h_1} + \cdots + \delta_{\hat{y}_l+h_l}$. For any distinct points x, y in $\Upsilon(\tilde{\Gamma}_n, \gamma', \Lambda')^{(1)}$, the norm $|x-y| > r$ because γ is in $\mathcal{K}(\tilde{\Gamma}_n)$ and (H). Moreover, since $\hat{\gamma}$ satisfies (H), any ball $B(x,R)$ where x is in $\mathcal{S}_{\Lambda'}(\tilde{\Gamma}_n)$ contains some points of $\Upsilon(\tilde{\Gamma}_n,\gamma',\Lambda')$. It is the same if x is not in $\mathcal{S}_{\Lambda'}(\tilde{\Gamma}_n)$ because $\tilde{\Gamma}_n$ is in $\mathcal{M}^\infty_{\mathcal{D}}(\mathcal{E})$ and (27). We deduce that any triangle X in $\Upsilon(\tilde{\Gamma}_n,\gamma,\Lambda')^{(3)}$ has the radius of $B(X)$ strictly lower than R. So, $\Upsilon(\tilde{\Gamma}_n,\gamma',\Lambda')$ is in $\mathcal{K}(\tilde{\Gamma}_n)$.

Now, we can begin the domination of $\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}(\mathcal{K}(\tilde{\Gamma}_n) - \mathcal{K}^\varepsilon(\tilde{\Gamma}_n))$.

$$\begin{aligned}
&\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}(\mathcal{K}(\tilde{\Gamma}_n) - \mathcal{K}^\varepsilon(\tilde{\Gamma}_n)) \\
&\quad = e^{-\lambda(S_{\Lambda'}(\tilde{\Gamma}_n)-\Lambda)} \sum_{k=0}^{k_0} \frac{1}{k!} \int \cdots \int_{(S_{\Lambda'}(\tilde{\Gamma}_n)-\Lambda)^k} \\
&\qquad \times \left[\int_{\mathcal{M}(\Lambda)} \mathbb{1}_{\mathcal{K}(\tilde{\Gamma}_n)-\mathcal{K}^\varepsilon(\tilde{\Gamma}_n)}(\delta_{x_1} + \cdots + \delta_{x_k} + \gamma')\pi_\Lambda(d\gamma') \right] dx_1 \cdots dx_k,
\end{aligned} \tag{36}$$

where k_0 is a bound for the number of points of elements in $\mathcal{K}(\tilde{\Gamma}_n)$. For $(x_i)_{1\le i\le k}$ in $\mathcal{S}_{\Lambda'}(\tilde{\Gamma}_n) - \Lambda$ and γ' in $\mathcal{M}(\Lambda)$ such that $\eta = \delta_{x_1} + \cdots + \delta_{x_k} + \gamma'$ is in $\mathcal{K}_{\Lambda'}(\tilde{\Gamma}_n)$, we note $\hat{\eta} = \delta\hat{y}_1 + \cdots + \delta\hat{y}_l$ defined above. Let us point out that $(\hat{y}_i)_{1\le i\le l}$ depends only on $x_1, \ldots, x_k$. So, we have from the definition of $\mathcal{K}^\varepsilon(\tilde{\Gamma}_n)$ the following domination

$$\begin{aligned}
&\int_{\mathcal{M}(\Lambda)} \mathbb{1}_{\mathcal{K}(\tilde{\Gamma}_n)-\mathcal{K}^\varepsilon(\tilde{\Gamma}_n)}(\delta_{x_1} + \cdots + \delta_{x_k} + \gamma')\pi_\Lambda(d\gamma') \\
&\quad \le \varepsilon \mathbb{1}_{\mathcal{K}(\tilde{\Gamma}_n)}(\delta_{x_1} + \cdots + \delta_{x_k} + \delta\hat{y}_1 + \cdots + \delta\hat{y}_l), \\
&\quad \le \varepsilon \left(\pi \left(\frac{\alpha}{3} \right)^2 \right)^{-l} \int_{B(\hat{y}_1, \frac{\alpha}{3})} \cdots \int_{B(\hat{y}_l, \frac{\alpha}{3})} \\
&\qquad \times \mathbb{1}_{\mathcal{K}(\tilde{\Gamma}_n)}(\delta_{x_1} + \cdots + \delta_{x_k} + \delta y_1 + \cdots + \delta y_l) dy_1 \ldots dy_l \\
&\quad \le \varepsilon \left(\pi \left(\frac{\alpha}{3} \right)^2 \right)^{-k_0} \sum_{l=0}^{k_0} \int \cdots \int_{(\mathcal{S}_{\Lambda'}(\tilde{\Gamma}_n))^l} \mathbb{1}_{\mathcal{K}(\tilde{\Gamma}_n)} \\
&\qquad \times (\delta_{x_1} + \cdots + \delta_{x_k} + \delta y_1 + \cdots + \delta y_l) dy_1 \ldots dy_l.
\end{aligned} \tag{37}$$

Compiling (36) and (37), we find

$$\begin{aligned}
&\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}(\mathcal{K}(\tilde{\Gamma}_n) - \mathcal{K}^\varepsilon(\tilde{\Gamma}_n)) \\
&\quad \le \varepsilon \left(\pi \left(\frac{\alpha}{3} \right)^2 \right)^{-k_0} e^{-\lambda(S_{\Lambda'}(\tilde{\Gamma}_n)-\Lambda)} \\
&\qquad \times \sum_{l=0}^{k_0} \sum_{k=0}^{k_0} \frac{1}{k!} \int \cdots \int_{(S_{\Lambda'}(\tilde{\Gamma}_n))^{k+l}} \mathbb{1}_{\mathcal{K}(\tilde{\Gamma}_n)} \\
&\qquad \times (\delta_{x_1} + \cdots + \delta_{x_k} + \delta y_1 + \cdots + \delta y_l) dx_1 \ldots dx_k dy_1 \ldots dy_l \\
&\quad \le \rho\varepsilon\pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}((\mathcal{K}_{\Lambda'}(\tilde{\Gamma}_n)),
\end{aligned} \tag{38}$$

where ρ is a strictly positive constant. Lemma 7 is proved. □

Now, we can prove the inequalities (26) which are expected. Thanks to Lemma 7 and inequality (34), equality (31) becomes

$$\int_{\mathcal{M}_{\mathcal{D}}^{\infty}(\mathcal{E})} f_\Lambda(\Gamma'^{(1)}) \Xi_{\Lambda'}(\tilde{\Gamma}_n, d\Gamma') \\ \leq \int_{\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)} f_\Lambda(\gamma) \mathbb{1}_{\mathcal{K}^\varepsilon(\tilde{\Gamma}_n)}(\gamma) \frac{e^{-E_{\Lambda'}(\Upsilon(\tilde{\Gamma}_n,\gamma,\Lambda'))}}{Z_{\Lambda'}(\tilde{\Gamma}_n)} \pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}(d\gamma) + \rho e^{2C_{\Lambda'}} \|f_\Lambda\|_\infty \varepsilon. \quad (39)$$

Since $\Lambda \subset \mathcal{S}_{\Lambda'}(\tilde{\Gamma}_n)$ and $\pi_{S_{\Lambda'}}(\tilde{\Gamma}_n) = \pi_\Lambda \, \pi_{S_{\Lambda'}(\tilde{\Gamma}_n)}$ we have

$$\int_{\mathcal{M}_{\mathcal{D}}^{\infty}(\mathcal{E})} f_\Lambda(\Gamma'^{(1)}) \Xi_{\Lambda'}(\tilde{\Gamma}_n, d\Gamma') \\ \leq \int_{\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)} f_\Lambda(\gamma) \mathbb{1}_{\mathcal{K}^\varepsilon(\Gamma)}(\gamma_\Lambda + \Gamma^{(1)}_{\mathcal{S}_{\Lambda'}(\Gamma)-\Lambda}) \\ \times \frac{e^{-E_{\Lambda'}(\Upsilon(\Gamma,\gamma_\Lambda+\Gamma^{(1)}_{\mathcal{S}_{\Lambda'}(\Gamma)-\Lambda},\Lambda'))}}{\int_{\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)} e^{-E_{\Lambda'}(\Upsilon(\Gamma,\gamma'_\Lambda+\Gamma^{(1)}_{\mathcal{S}_{\Lambda'}(\Gamma)-\Lambda},\Lambda'))} \pi_\Lambda(d\gamma'_\Lambda)} \pi_\Lambda(d\gamma_\Lambda) \Xi_{\Lambda'}(\tilde{\Gamma}_n, d\Gamma) \\ + \rho e^{2C_{\Lambda'}} \|f_\Lambda\|_\infty \varepsilon.$$

When $\gamma_\Lambda + \Gamma^{(1)}_{\mathcal{S}_{\Lambda'}(\Gamma)-\Lambda}$ is in $\mathcal{K}^\varepsilon(\Gamma)$ then $\int_{\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)} e^{-E_{\Lambda'}(\Upsilon(\Gamma,\gamma'_\Lambda+\Gamma^{(1)}_{\mathcal{S}_{\Lambda'}(\Gamma)-\Lambda},\Lambda'))} \pi_\Lambda(d\gamma') \geq \varepsilon e^{-C_{\Lambda'}}$. So,

$$\int_{\tilde{\Omega}} \int_{\mathcal{M}_{\mathcal{D}}^{\infty}(\mathcal{E})} f_\Lambda(\Gamma'^{(1)}) \Xi_{\Lambda'}(\tilde{\Gamma}_n, d\Gamma') d\tilde{P} \\ \leq \int_{\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)} f_\Lambda(\gamma) \frac{e^{2C_{\Lambda'}}}{\varepsilon} \pi_\Lambda(d\gamma) + \rho e^{2C_{\Lambda'}} \|f_\Lambda\|_\infty \varepsilon. \quad (40)$$

From (40) and (30), we deduce

$$\int_{\mathcal{M}(\mathcal{E})} f_\Lambda(\Gamma^{(1)}) \mu(d\Gamma) \leq \int_{\mathcal{M}_{\mathcal{D}}(\mathbb{R}^2)} f_\Lambda(\gamma) \frac{e^{2C_{\Lambda'}}}{\varepsilon} \pi_\Lambda(d\gamma) + \rho e^{2C_{\Lambda'}} \|f_\Lambda\|_\infty \varepsilon,$$

which is inequality (26) with the precise constants. Thanks to a monotone class argument, for every A in the σ-fields of $\mathcal{M}(\Lambda)$, we have $\mu^{(1)}(A) \leq e^{2C_{\Lambda'}}(\frac{\pi_\Lambda(A)}{\varepsilon} + \rho\varepsilon)$. If $\pi_\Lambda(A) = 0$ and if ε goes to 0, we have $\mu(A) = 0$. So, μ is locally absolutely continuous with respect to π_Λ. Lemma 6 is proved. □

Now, using Lemma 6 and the lemmas similar to 1 and 4, we prove Theorem 2. □

Let us suppose we want to construct an infinite random Delaunay tessellation for which each angle is bigger than a fixed angle α_0. We propose two approaches. In the first, we can consider an interaction potential which is equal to plus infinity if, and only if, a triangle has a minimal angle lower than α_0. With our method, it is possible to construct a Gibbs Delaunay tessellation with this potential. The proof is exactly the same as in Theorems 1 and 2. We just have to prove that the Lemmas 1, 3 and 4 are still true. The calculus are long and complicated and we only have the result for α_0 small enough. In a second easier approach, we use Theorem 2 and fix r and R such that any triangle, with the edges larger than r and the radius of the circumscribed ball lower than R, has necessary the minimal angle bigger than α_0 (for example, $r = 1$ and $R = \frac{1}{2\sin(\alpha_0)}$). In Theorem 2, the only assumption on r and R is $R > r$. So, the only condition on α_0 is $\alpha_0 < \frac{\pi}{6}$.

To finish this paper in conclusion, we would like to say that it is possible to extend our results to many different models of tessellations. It is however difficult to give general results because the interaction depends very strongly on the nature of the tessellations. It is necessary to adapt the results to the models. However, for the Voronoi tessellations, the results can be transposed directly thanks to the classical duality.

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